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Global existence result for thermoviscoelastic problems with hysteresis

Dedicated to the memory of M. Schatzman

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Abstract

We consider viscoelastic solids undergoing thermal expansion and exhibiting hysteresis effects due to plasticity or phase transformations. Within the framework of generalized standard solids, the problem is described in a 3D setting by the momentum equilibrium equation, the flow rule describing the dependence of the stress on the strain history, and the heat transfer equation. Under appropriate regularity assumptions on the data, a local existence result for this thermodynamically consistent system is established, by combining existence results for ordinary differential equations in Banach spaces with a fixed-point argument. Then global estimates are obtained by using both the classical energy estimate and more specific techniques for the heat equation introduced by Boccardo and Gallouët. Finally a global existence result is derived.

1 Introduction

We consider quasi-static boundary-value problems with internal variables which model viscoplasticity or phase-transformations in shape memory alloys. In such models, the *momentum equilibrium equation* is coupled to a *flow rule*, which is a constitutive relation describing the dependence of the stress on the strain history. In the framework of generalized standard materials (see [HaN75]), the unknowns are the *displacement* u and an *internal variable* z and the flow rule consists in a differential inclusion involving a dissipation potential. Such problems have been intensively studied during the last decade and existence results have been obtained by using either classical results for maximal monotone operators (see [AlC04]) or specific techniques for rate-independent processes introduced in [MiT04, Mie05] and later on developed in [FrM06, MiR06, MiR07, Mie07, MiP07, MRS08].

In this paper we are interested by the coupling of these problems with thermal effects. Indeed, plasticity or phase transformations are inelastic processes leading to energy dissipation; the temperature of the material increases and the change of temperature have also some influence on the mechanical behavior. Hence thermal effects can not be avoided. Many references on this topic are available in the engineering literature, where several computational methods are employed to obtain approximate solutions (see for instance [AdSCC99, SrZ99, RR*00, CaB04, HWR05]). More recently, especially suited numerical schemes have been proposed and a rigorous proof of their convergence is established to so called *energetic solutions*, which solve the problem only in a weak form (see [BaR08, Rou10, BaR11]).

The aim of this paper is thus to prove a global existence result for such problems in a more classical sense by using a fixed-point argument. The model that we consider is based on the Helmholtz free energy $W(\mathrm{e}(u), z, \theta)$, depending on the *infinitesimal strain tensor* $\mathrm{e}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ for the displacement $u : \Omega\times(0,T) \to \mathbb{R}^3$, where $(\cdot)^{\mathsf{T}}$ denotes the transpose of a tensor, the *internal variable* $z : \Omega\times(0,T) \to \mathcal{Z}$, where $\mathcal{Z}$ is a finite dimensional real vector space, and the *temperature* $\theta : \Omega\times(0,T) \to \mathbb{R}$. For simplicity, we will omit any dependence on the material point $x \in \Omega$ and $t \in [0,T]$ with $T > 0$. We assume that W can be decomposed as follows

$$W(\mathrm{e}(u), z, \theta) \stackrel{\text{def}}{=} W_1(\mathrm{e}(u), z) - W_0(\theta) + \theta W_2(\mathrm{e}(u)). \tag{1.1}$$

The above decomposition ensures that entropy separates the thermal and mechanical variables (see (2.8)). Note that the last term in (1.1) will lead to coupling terms in the momentum equilibrium equation and in the heat equation but not in the flow rule. For a more general setting the reader is referred to [PaP11]. We make the assumptions of small deformations. The problem is thus described by the following system

$$-\operatorname{div}(\sigma_{\rm el}+\mathbb{L}{\rm e}(\dot u)) = \ell, \quad \sigma_{\rm el} \stackrel{\text{def}}{=} {\rm D}_{{\rm e}(u)}W({\rm e}(u),z,\theta), \tag{1.2a}$$

$$\partial\Psi(\dot z) + \mathbb{M}\dot z + \sigma_{\rm in} \ni 0, \quad \sigma_{\rm in} \stackrel{\text{def}}{=} {\rm D}_z W({\rm e}(u),z,\theta), \tag{1.2b}$$

$$c(\theta)\dot\theta - \operatorname{div}(\kappa({\rm e}(u),z,\theta)\nabla\theta) = \mathbb{L}{\rm e}(\dot u){:}{\rm e}(\dot u) + \theta\partial_t W_2({\rm e}(u)) + \Psi(\dot z) + \mathbb{M}\dot z.\dot z. \tag{1.2c}$$

We have denoted the dissipation potential by Ψ, which is assumed to be positively homogeneous of degree 1, i.e., $\Psi(\gamma z) = \gamma\Psi(z)$ for all $\gamma \geq 0$. This assumption is commonly used in modeling hysteresis effect in mechanics (see [MiT04, Mie05]). The viscosity tensors are denoted by $\mathbb{L}$ and $\mathbb{M}$, $c(\theta)$ is the *heat capacity* and $\kappa({\rm e}(u),z,\theta)$ is the *conductivity*. We have used above the following notations: $\dot{(\,)}$, ${\rm D}_z^i$ and ∂ denote the time derivative $\frac{\partial}{\partial t}$, the i-th derivative with respect to z and the subdifferential in the sense of convex analysis (for more details see [Bre73]). Moreover ${\rm e}_1{:}{\rm e}_2$ and $z_1.z_2$ denote the inner product of ${\rm e}_1$ and ${\rm e}_2$ in $\mathbb{R}^{3\times 3}_{\text{sym}}$ and z_1 and z_2 in $\mathcal{Z}$. Observe that (1.2a), (1.2b) and (1.2c) are usually called the momentum equilibrium equation, the flow rule and the heat-transfer equation, respectively.

The paper is organized as follows. We justify the thermodynamic consistency of our model and we present some illustrative examples in Section 2. In Section 3 the mathematical formulation of the problem in terms of displacement, internal variable and temperature is given. Then, using the classical enthalpy transformation, another formulation is obtained. In Section 4, we establish existence and regularity results for the system composed by the momentum equilibrium equation and the flow rule for a given temperature θ, and in Section 5, existence and regularity results for the enthalpy equation for any given right-hand side are recalled. Then in Section 6, a local existence result for the coupled problem follows by using a fixed-point argument. Finally, global energy estimates are obtained in Section 7, leading to a global existence result for the system (1.2).

2 Mechanical model

2.1 Thermodynamic consistency

Starting from the Helmholtz free energy W, let us introduce the *internal energy* $W_{\rm in}$ defined by

$$W_{\rm in}({\rm e}(u),z,\theta) \stackrel{\text{def}}{=} W({\rm e}(u),z,\theta) + \theta s. \tag{2.1}$$

and the specific *entropy* s by using the so-called Gibb's relation

$$s \stackrel{\text{def}}{=} -{\rm D}_\theta W({\rm e}(u),z,\theta). \tag{2.2}$$

Then the *entropy equation*

$$\theta\dot s + \operatorname{div}(j) = \xi, \tag{2.3}$$

gives some balance between the *heat flux* j and the heat production due to the *dissipation rate* ξ. In particular, we have

$$\xi \stackrel{\text{def}}{=} \mathbb{L}{\rm e}(\dot u){:}{\rm e}(\dot u) + \mathbb{M}\dot z.\dot z + \Psi(\dot z) \geq 0.$$

Furthermore we assume that the Fourier's law for the temperature holds, namely

$$j \stackrel{\text{def}}{=} -\kappa({\rm e}(u),z,\theta)\nabla\theta.$$

We can check now that the second law of thermodynamics holds if $\theta > 0$. Indeed, under the classical assumption that the system is thermally isolated, it is possible to divide (2.3) by θ and due to the Green's formula, we get

$$\begin{aligned}\int_\Omega \dot{s}\,\mathrm{d}x &= \int_\Omega \tfrac{\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)}{\theta}\,\mathrm{d}x + \int_\Omega \tfrac{\mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})+\mathbb{M}\dot{z}.\dot{z}+\Psi(\dot{z})}{\theta}\,\mathrm{d}x\\ &= \int_\Omega \tfrac{\kappa(\mathrm{e}(u),z,\theta)\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x + \int_\Omega \tfrac{\mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})+\mathbb{M}\dot{z}.\dot{z}+\Psi(\dot{z})}{\theta}\,\mathrm{d}x \geq 0.\end{aligned}$$

We differentiate $W_{\mathrm{in}}(\mathrm{e}(u),z,\theta)$ with respect to time. By using the chain rule and (2.1), we obtain

$$\dot{W}_{\mathrm{in}}(\mathrm{e}(u),z,\theta) = \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\theta):\mathrm{e}(\dot{u}) + \mathrm{D}_z W(\mathrm{e}(u),z,\theta).\dot{z} + \theta\dot{s}. \tag{2.4}$$

We integrate (2.4) over Ω, then we use the Green's formula and (2.3), we find

$$\begin{aligned}\int_\Omega \dot{W}_{\mathrm{in}}(\mathrm{e}(u),z,\theta)\,\mathrm{d}x &= \int_\Omega \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\theta):\mathrm{e}(\dot{u})\,\mathrm{d}x + \int_\Omega \mathrm{D}_z W(\mathrm{e}(u),z,\theta).\dot{z}\,\mathrm{d}x\\ &+ \int_\Omega (\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)+\mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})+\mathbb{M}\dot{z}.\dot{z}+\Psi(\dot{z}))\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

On the one hand, we multiply (1.2a) by $\dot{u}$, and we integrate this expression over Ω to get

$$\int_\Omega \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\theta):\mathrm{e}(\dot{u})\,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})\,\mathrm{d}x = \int_\Omega \ell\cdot\dot{u}\,\mathrm{d}x. \tag{2.6}$$

On the other hand, we multiply (1.2b) by $\dot{z}$ and the definition of the subdifferential $\partial\Psi(\dot{z})$ leads to the following equality

$$\int_\Omega \mathrm{D}_z W(\mathrm{e}(u),z,\theta).\dot{z}\,\mathrm{d}x + \int_\Omega \mathbb{M}\dot{z}.\dot{z}\,\mathrm{d}x + \int_\Omega \Psi(\dot{z})\,\mathrm{d}x = 0. \tag{2.7}$$

We use (2.6) and (2.7) into (2.5), we obtain

$$\int_\Omega \dot{W}_{\mathrm{in}}(\mathrm{e}(u),z,\theta)\,\mathrm{d}x = \int_\Omega \ell\cdot\dot{u}\,\mathrm{d}x + \int_{\partial\Omega} \kappa(\mathrm{e}(u),z,\theta)\nabla\theta\cdot\eta\,\mathrm{d}x.$$

This means that the total energy balance can be expressed in terms of the internal energy, which is the sum of power of external load and heat. Finally, from (2.2), we may deduce that

$$s \overset{\mathrm{def}}{=} \mathrm{D}_\theta W_0(\theta) - W_2(\mathrm{e}(u)), \tag{2.8}$$

and

$$W_{\mathrm{in}}(\mathrm{e}(u),z,\theta) \overset{\mathrm{def}}{=} W_1(\mathrm{e}(u),z,\theta) + \theta\mathrm{D}_\theta W_0(\theta) - W_0(\theta). \tag{2.9}$$

Inserting (2.8) into (2.3), we may deduce the heat-transfer equation (1.2c) with the heat capacity given by $c(\theta) = \theta\mathrm{D}^2_\theta W_0(\theta)$.

2.2 Examples of admissible constitutive models

Let us illustrate our setting by several examples coming from plasticity theory or three dimensional modelization of phase transformations in shape memory alloys. For each of the following examples, the assumptions introduced in Section 3 are satisfied, and thus the global existence result for the presented systems follows from the abstract result obtained in the next sections.

2.2.1 Thermoviscoplasticity

We consider a viscoelastic solid in Kelvin-Voigt rheology involving plasticity and undergoing isotropic thermal expansion. The Helmholtz free energy is given by (1.1) with

$$W_1(\mathrm{e}(u),z) \overset{\text{def}}{=} \tfrac12\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){:}(\mathrm{e}(u){-}\mathbb{B}z){+}H(z) \quad \text{and} \quad W_2(\mathrm{e}(u)) \overset{\text{def}}{=} \alpha\mathrm{tr}(\mathrm{e}(u)).$$

Here the internal variable z belongs to a finite dimensional real vector space $\mathcal{Z}$ and $\mathbb{B}$ is a linear mapping from $\mathcal{Z}$ to the space of deviatoric 3×3 tensors which associates to the internal variable z the plastic strain $\mathrm{e}_{\text{plast}} \overset{\text{def}}{=} \mathbb{B}z$, $\alpha\ge0$ is the thermal expansion coefficient and H is the hardening functional. Then the system (1.2) can be rewritten as

$$\begin{aligned}
&-\mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u)) = \ell,\\
&\partial\Psi(\dot z)+\mathbb{M}\dot z-\mathbb{B}^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)+\mathrm{D}_zH(z)\ni 0,\\
&c(\theta)\dot\theta-\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u)+\alpha\theta\mathrm{tr}(\mathrm{e}(\dot u))+\Psi(\dot z)+\mathbb{M}\dot z.\dot z.
\end{aligned}$$

Note that I is the identity matrix and $\mathbb{B}^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)$ is the linear mapping defined on $\mathcal{Z}$ by $\widetilde z\mapsto\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){:}\mathbb{B}\widetilde z$. We should confess here a lack of consistency in the notations. Indeed, we have $\mathbb{B}\in\mathcal{L}(\mathcal{Z};\mathbb{R}^{3\times3}_{\mathsf{dev}})$, thus $\mathbb{B}^{\mathsf{T}}\in\mathcal{L}(\mathbb{R}^{3\times3}_{\mathsf{dev}};\mathcal{Z})$ while $\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)\in\mathbb{R}^{3\times3}_{\mathsf{sym}}$. Hence $\mathbb{B}^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)$ has to be understood as $\mathbb{B}^{\mathsf{T}}\mathrm{proj}_{\mathbb{R}^{3\times3}_{\mathsf{dev}}}(\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)$ where $\mathrm{proj}_{\mathbb{R}^{3\times3}_{\mathsf{dev}}}$ is the projection on $\mathbb{R}^{3\times3}_{\mathsf{dev}}$ relatively to the inner product of $\mathbb{R}^{3\times3}_{\mathsf{sym}}$.

Let us emphasize that several viscoplastic models fit this general description. Indeed the Melan-Prager model corresponds to a kinematic linear hardening, i.e., a hardening functional H given by

$$\forall z\in\mathcal{Z}:\ H(z)=\tfrac12 Lz{\cdot}z$$

with a symmetric positive definite tensor $L\in\mathcal{L}(\mathcal{Z};\mathcal{Z})$ while the Prandtl-Reuss model corresponds to $H\equiv0$. The choice of a rate-dependent plasticity flow rule (due to the viscous term $\mathbb{M}\dot z$ in (1.2b)) is physically meaningful since we expect more dissipation in case of rapid change in the plastic strain (see also [DDM06] for a more detailed discussion when $H\equiv0$).

2.2.2 Phase transformations in shape-memory alloys

In order to gather in the same description both the three-dimensional macroscopic phenomenological model for shape-memory polycrystalline material introduced by Souza et al [SMZ98] and later addressed and extended by Auricchio et al [AuP02, AuP04], and so called mixture models, we consider

$$W_1(\mathrm{e}(u),z) \overset{\text{def}}{=} \tfrac12\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){:}(\mathrm{e}(u){-}\mathbb{B}z){+}H(z) \quad \text{and} \quad W_2(\mathrm{e}(u)) \overset{\text{def}}{=} \alpha\mathrm{tr}(\mathrm{e}(u)).$$

Once again the internal variable z belongs to a finite dimensional real vector space $\mathcal{Z}$, $\alpha\ge0$ is the thermal expansion coefficient and H is the hardening functional. But now $\mathbb{B}$ is an affine mapping from $\mathcal{Z}$ to the space of 3×3 deviatoric tensors and it can be decomposed as

$$\forall z\in\mathcal{Z}:\ \mathbb{B}z=\mathbb{B}_0z+B_1,$$

with $\mathbb{B}_0\in\mathcal{L}(\mathcal{Z};\mathbb{R}^{3\times3}_{\mathsf{dev}})$ and $B_1\in\mathbb{R}^{3\times3}_{\mathsf{dev}}$. The problem (1.2) can be rewritten as

$$\begin{aligned}
&-\mathrm{div}(\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u)) = \ell,\\
&\partial\Psi(\dot z)+\mathbb{M}\dot z-\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z)+\mathrm{D}_zH(z)\ni 0,\\
&c(\theta)\dot\theta-\mathrm{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u)+\alpha\theta\mathrm{tr}(\mathrm{e}(\dot u))+\Psi(\dot z)+\mathbb{M}\dot z.\dot z.
\end{aligned}$$

In the Souza-Auricchio model, we have $\mathcal{Z} = \mathbb{R}^{3\times3}_{\text{dev}}$, $\mathbb{B} \overset{\text{def}}{=} \mathrm{I}_{\mathcal{Z}}$ and the variable z describes the inelastic part of the deformation coming from the martensitic phase transformations. Furthermore the hardening functional H_{SA} takes the following form:

$$H_{\mathrm{SA}}(z) \overset{\text{def}}{=} c_1|z| + c_2|z|^2 + \chi(z),$$

where $\chi : \mathbb{R}^{3\times3}_{\text{dev}} \to [0, +\infty]$ denotes the indicator function of the ball $\{z \in \mathbb{R}^{3\times3}_{\text{dev}} : \ |z_\nu| \le c_3\}$ and the coefficients c_i, $i = 1, 2, 3$, are positive real numbers. Note that c_1 is an activation threshold for initiation of martensitic phase transformations, c_2 measures the occurrence of hardening with respect to the internal variable z, c_3 represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants. We should regularize the hardening functional as in [MiP07] and we replace $H_{\mathrm{SA}}(z)$ by $H^\delta_{\mathrm{SA}}(z)$ by

$$H^\delta_{\mathrm{SA}}(z) \overset{\text{def}}{=} c_1\sqrt{\delta^2+|z|^2} + \tfrac{c_2}{2}|z|^2 + \tfrac{((|z|-c_3)_+)^4}{\delta(1+|z|^2)},$$

where $0 < \delta \ll 1$.

In the mixture models, we have $\mathcal{Z} = \mathbb{R}^{N-1}$, where N is the total number of phases, including austenite and all the variants of martensite (hence $N \ge 2$). The components of z and $z_N = 1 - \sum_{k=1}^{N-1} z_k$ can be interpreted as phase fractions and $\mathbb{B}z$ is the effective transformation strain of the mixture given by

$$\mathbb{B}z \overset{\text{def}}{=} \sum_{k=1}^{N-1} z_k E_k + \Big(1 - \sum_{k=1}^{N-1} z_k\Big) E_N,$$

where E_k is the transformation strain of the phase k. The hardening functional H_{mixt} is the sum of a smooth part $w(z)$ and the indicator function of $[0, 1]^{N-1}$ (see [MiT99, Mie00, HaG02, GMH02, MTL02, GHH07]).

This non-smooth part is regularized in the same way as previously and we consider a regularized hardening functional H^δ_{mixt} given by

$$H^\delta_{\mathrm{mixt}}(z) \overset{\text{def}}{=} w(z) + \sum_{k=1}^{N-1} \tfrac{((-z_k)_+)^4 + ((z_k-1)_+)^4}{\delta(1+|z_k|^2)}, \quad 0 < \delta \ll 1.$$

In both models, the rate-dependent flow rule is once again physically meaningful since we expect more dissipation in case of rapid change in the crystalline structure of the material. Furthermore, it allows us to avoid the usual gradient regularization for the internal variable.

3 Mathematical formulation

We consider a reference configuration $\Omega \subset \mathbb{R}^3$. We assume that Ω is a bounded domain such that $\partial\Omega$ is of class C^2. We will denote by $\mathbb{R}^{3\times3}_{\text{sym}}$ (respectively $\mathbb{R}^{3\times3}_{\text{dev}}$) the space of symmetric (respectively deviatoric) 3×3 tensors endowed with the natural scalar product $v{:}w \overset{\text{def}}{=} \mathrm{tr}(v^{\mathsf{T}} w)$ and the corresponding norm $|v|^2 \overset{\text{def}}{=} v{:}v$ for all $v, w \in \mathbb{R}^{3\times3}_{\text{sym}}$. In particular, we assume that

$$W_1(\mathrm{e}(u), z) \overset{\text{def}}{=} \tfrac{1}{2}\mathbb{E}(\mathrm{e}(u) - \mathbb{B}z){:}(\mathrm{e}(u) - \mathbb{B}z) + H(z) \quad \text{and} \quad W_2(\mathrm{e}(u)) \overset{\text{def}}{=} \alpha\mathrm{tr}(\mathrm{e}(u)),$$

where $\alpha \ge 0$ is the thermal expansion coefficient, $\mathbb{E}$ denotes the elastic tensor, H the hardening functional and $\mathbb{B}$ is an affine mapping from the finite dimensional vectorial space $\mathcal{Z}$ to $\mathbb{R}^{3\times3}_{\text{dev}}$. More precisely, $\mathbb{B}$ is decomposed as follows

$$\forall z \in \mathcal{Z} : \ \mathbb{B}z \overset{\text{def}}{=} \mathbb{B}_0 z + B_1, \tag{3.1}$$

where $\mathbb{B}_0 \in \mathcal{L}(\mathcal{Z}, \mathbb{R}^{3\times 3}_{\mathsf{dev}})$ and $B_1 \in \mathbb{R}^{3\times 3}_{\mathsf{dev}}$. Given a function $\ell : \Omega\times(0,T) \to \mathbb{R}^3$, we look for a displacement $u : \Omega\times(0,T) \to \mathbb{R}^3$, an internal variable $z : \Omega\times(0,T) \to \mathcal{Z}$ and a temperature $\theta : \Omega\times(0,T) \to \mathbb{R}$ satisfying the following system:

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z)+\alpha\theta\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{3.2a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z) + \mathrm{D}_z H(z) \ni 0, \tag{3.2b}$$

$$c(\theta)\dot{\theta} - \operatorname{div}(\kappa(\mathrm{e}(u), z, \theta)\nabla\theta) = \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})) + \Psi(\dot{z}) + \mathbb{M}\dot{z}.\dot{z}, \tag{3.2c}$$

together with initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \theta(\cdot,0) = \theta^0, \tag{3.3}$$

and boundary conditions

$$u_{|\partial\Omega} = 0, \quad \kappa\nabla\theta{\cdot}\eta_{|\partial\Omega} = 0, \tag{3.4}$$

where η denotes the outward normal to the boundary $\partial\Omega$ of Ω. We reformulate now the original problem (3.2) in terms of enthalpy instead of temperature by using the so-called enthalpy transformation defined as follows

$$g(\theta) = \vartheta \overset{\mathsf{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s. \tag{3.5}$$

Note that g is the unique primitive of the function c, which is supposed to be continuous, such that $g(0) = 0$. Furthermore, we will assume that for all $s \geq 0$, $c(s) \geq c^c > 0$ where c^c is a constant. Hence we deduce that g is a bijection from $[0,\infty)$ into $[0,\infty)$. We define

$$\zeta(\vartheta) \overset{\mathsf{def}}{=} \begin{cases} g^{-1}(\vartheta) & \text{if } \vartheta \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.6a}$$

$$\kappa^c(\mathrm{e}(u), z, \vartheta) \overset{\mathsf{def}}{=} \tfrac{\kappa(\mathrm{e}(u), z, \zeta(\vartheta))}{c(\zeta(\vartheta))}, \tag{3.6b}$$

where g^{-1} is the inverse of g. For more details on the enthalpy transformation, the reader is referred to [Rou09] and the references therein. Then the system (3.2) is rewritten as follows

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z)+\alpha\zeta(\vartheta)\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{3.7a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z) + \mathrm{D}_z H(z) \ni 0, \tag{3.7b}$$

$$\dot{\vartheta} - \operatorname{div}(\kappa^c(\mathrm{e}(u), z, \vartheta)\nabla\vartheta) = \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \alpha\zeta(\vartheta)\mathrm{tr}(\mathrm{e}(\dot{u})) + \Psi(\dot{z}) + \mathbb{M}\dot{z}.\dot{z}, \tag{3.7c}$$

with boundary conditions

$$u_{|\partial\Omega} = 0, \quad \kappa^c\nabla\vartheta{\cdot}\eta_{|\partial\Omega} = 0, \tag{3.8}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \quad \vartheta(\cdot,0) = \vartheta^0 = g(\theta^0). \tag{3.9}$$

Usually, the identity (3.7c) is called the enthalpy equation. Since we have assumed that $\partial\Omega$ is of class C^2, Korn's inequality holds, i.e., we have

$$\exists C^{\mathsf{Korn}} > 0\ \forall u \in \mathrm{H}^1_0(\Omega):\ \|\mathrm{e}(u)\|^2_{\mathrm{L}^2(\Omega)} \geq C^{\mathsf{Korn}}\|u\|^2_{\mathrm{H}^1(\Omega)}. \tag{3.10}$$

The reader is referred to [KoO88, DuL76] for further details.

We introduce now the assumptions on the dissipation potential Ψ, on the hardening function H, and on the data $\mathbb{E}$, $\mathbb{L}$, $\mathbb{M}$, ℓ, $c \stackrel{\text{def}}{=} c(\theta)$ and $\kappa \stackrel{\text{def}}{=} \kappa(\mathrm{e}(u), z, \theta)$.

The dissipation potential Ψ is assumed to be positively homogeneous of degree 1 and satisfies the triangle inequality, i.e., we have

$$\forall \gamma \geq 0 \; \forall z \in \mathcal{Z} : \; \Psi(\gamma z) = \gamma \Psi(z), \tag{3.11a}$$

$$\exists C^{\Psi} > 0 \; \forall z \in \mathcal{Z} : \; 0 \leq \Psi(z) \leq C^{\Psi}|z|, \tag{3.11b}$$

$$\forall z_1, z_2 \in \mathcal{Z} : \; \Psi(z_1+z_2) \leq \Psi(z_1) + \Psi(z_2). \tag{3.11c}$$

Observe that (3.11a), (3.11b) and (3.11c) imply that Ψ is convex and continuous. We assume that the hardening functional H belongs to $\mathrm{C}^2(\mathcal{Z};\mathbb{R})$ and satisfies the following inequalities

$$\exists c^H, \widetilde{c}^H \geq 0 \; \forall z \in \mathcal{Z} : \; H(z) \geq c^H |z|^2 - \widetilde{c}^H, \tag{3.12a}$$

$$\exists C^H_{zz} > 0 \; \forall z \in \mathcal{Z} : \; |\mathrm{D}^2_z H(z)| \leq C^H_{zz}. \tag{3.12b}$$

Note that (3.12b) leads to

$$\exists C^H_z > 0 \; \forall z \in \mathcal{Z} : \; |\mathrm{D}_z H(z)| \leq C^H_z (1+|z|), \quad |H(z)| \leq C^H_z (1+|z|^2). \tag{3.13}$$

The elastic tensor $\mathbb{E} : \Omega \to \mathcal{L}(\mathbb{R}^{3\times3}_{\mathrm{sym}}, \mathbb{R}^{3\times3}_{\mathrm{sym}})$ is a symmetric positive definite operator such that

$$\exists c^{\mathbb{E}} > 0 \; \forall z \in \mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}}) : \; c^{\mathbb{E}} \|z\|^2_{\mathrm{L}^2(\Omega)} \leq \int_\Omega \mathbb{E}z{:}z \,\mathrm{d}x, \tag{3.14a}$$

$$\forall i,j,k = 1,2,3 : \; \mathbb{E}(\cdot), \frac{\partial \mathbb{E}_{i,j}(\cdot)}{\partial x_k} \in \mathrm{L}^\infty(\Omega). \tag{3.14b}$$

We suppose that $\mathbb{L}$ and $\mathbb{M}$ are symmetric positive definite tensors. This implies that

$$\exists c^{\mathbb{L}}, C^{\mathbb{L}} > 0 \; \forall \mathrm{e} \in \mathbb{R}^{3\times3}_{\mathrm{sym}} : \; c^{\mathbb{L}}|\mathrm{e}|^2 \leq \mathbb{L}\mathrm{e}{:}\mathrm{e} \leq C^{\mathbb{L}}|\mathrm{e}|^2, \tag{3.15a}$$

$$\exists c^{\mathbb{M}}, C^{\mathbb{M}} > 0 \; \forall z \in \mathcal{Z} : \; c^{\mathbb{M}}|z|^2 \leq \mathbb{M}z.z \leq C^{\mathbb{M}}|z|^2. \tag{3.15b}$$

We consider that ℓ is an external loading satisfying

$$\ell \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)). \tag{3.16}$$

Finally, the heat capacity c and the conductivity κ^c satisfy

$$c : [0,\infty) \to [0,\infty) \text{ is continuous,} \tag{3.17a}$$

$$\exists \beta_1 \geq 2 \; \exists c^c > 0 \; \forall \theta \geq 0 : \; c^c (1+\theta)^{\beta_1 - 1} \leq c(\theta), \tag{3.17b}$$

$$\kappa^c : \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathcal{Z} \times \mathbb{R} \to \mathbb{R}^{3\times3}_{\mathrm{sym}} \text{ is continuous,} \tag{3.17c}$$

$$\exists c^{\kappa^c} > 0 \; \forall (e,z,\vartheta) \in \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathcal{Z} \times \mathbb{R} \; \forall v \in \mathbb{R}^3 : \; \kappa^c(e,z,\vartheta) v \cdot v \geq c^{\kappa^c}|v|^2, \tag{3.17d}$$

$$\exists C^{\kappa^c} > 0 \; \forall (e,z,\vartheta) \in \mathbb{R}^{3\times3}_{\mathrm{sym}} \times \mathcal{Z} \times \mathbb{R} : \; |\kappa^c(e,z,\vartheta)| \leq C^{\kappa^c}. \tag{3.17e}$$

Let us give now some indications about the proof strategy. First, we establish a local existence result for the coupled problem (3.7)–(3.9) by using a fixed point argument. More precisely, for any given $\widetilde{\vartheta}$, we define $\theta \stackrel{\text{def}}{=} \zeta(\widetilde{\vartheta})$ and we solve the system composed by the momentum equilibrium equation and the flow rule (3.2a)–(3.2b), then we solve the enthalpy equation (3.7c) with $\kappa^c \stackrel{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \zeta(\widetilde{\vartheta}))$. This allows us to define a mapping

$$\phi^{\widetilde{\vartheta},\vartheta} : \; \widetilde{\vartheta} \to \vartheta.$$

Our goal consists to prove that this mapping satisfies the assumptions of Schauder's fixed point theorem. To this aim, we consider a given $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ with $\bar{p} \geq 1$ and $\bar{q} \geq 1$. We define $\theta \overset{\text{def}}{=} \zeta(\widetilde{\vartheta})$. Since ζ is a Lipschitz continuous mapping from $\mathbb{R}$ to $\mathbb{R}$, it follows that the mapping

$$\begin{aligned} \phi^{\widetilde{\vartheta},\theta}:\ \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega)) &\to \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega)), \\ \widetilde{\vartheta} &\mapsto \theta = \zeta(\widetilde{\vartheta}), \end{aligned}$$

is also Lipschitz continuous. On the other hand, the inequality (3.17b) implies that

$$\forall \theta \in [0,\infty):\ \tfrac{c^c}{\beta_1}((1+\theta)^{\beta_1}-1) \overset{\text{def}}{=} g_1(\theta) \leq g(\theta).$$

Thus we obtain

$$\forall \vartheta \in [0,\infty):\ 0 \leq \zeta(\vartheta) \leq \zeta_1(\vartheta) \overset{\text{def}}{=} g_1^{-1}(\vartheta),$$

and

$$\forall \vartheta \in \mathbb{R}:\ |\zeta(\vartheta)| \leq \big(\tfrac{\beta_1}{c^c}\max(\vartheta,0)+1\big)^{\frac{1}{\beta_1}} - 1.$$

Clearly, we may infer that

$$\forall \beta \in [1,\beta_1]\ \forall \vartheta \in \mathbb{R}:\ |\zeta(\vartheta)| \leq \big(\tfrac{\beta_1}{c^c}\max(\vartheta,0)+1\big)^{\frac{1}{\beta}} - 1 \leq \big(\tfrac{\beta_1}{c^c}\max(\vartheta,0)\big)^{\frac{1}{\beta}}. \tag{3.18}$$

Therefore for all $\beta \in [1,\beta_1]$ and for all $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we have $\theta = \zeta(\widetilde{\vartheta}) \in \mathrm{L}^{\beta\bar{q}}(0,T;\mathrm{L}^{\beta\bar{p}}(\Omega))$ with

$$\|\theta\|_{\mathrm{L}^{\beta\bar{q}}(0,T;\mathrm{L}^{\beta\bar{p}}(\Omega))} \leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta}} \|\widetilde{\vartheta}\|^{\frac{1}{\beta}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))}.$$

We assume that $\bar{q} > 2$ and $\bar{p} = 2$ in the sequel. Note that if there is not any confusion, we will use simply the notation $\mathrm{X}(\Omega)$ instead of $\mathrm{X}(\Omega;\mathrm{Y})$ where X is a functional space and Y is a finite dimensional real vector space. We also use the notation $\mathcal{Q}_\tau \overset{\text{def}}{=} \Omega \times (0,\tau)$ with $\tau \in [0,T]$.

4 Existence, uniqueness and regularity results for the system composed by the momentum equilibrium equation and the flow rule

We establish in this section existence and uniqueness results for the system composed by the momentum equilibrium equation and the flow rule (3.2a)–(3.2b) when $\theta = \zeta(\widetilde{\vartheta})$ is a given data in a bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ with $q \in [\bar{q},\beta_1\bar{q}]$ and $p \in [\bar{p},\min(\beta_1\bar{p},6)]$. More precisely, we prove that (P_{uz})

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{4.1a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - \mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}\mathbb{B}z) + \mathrm{D}_z H(z) \ni 0, \tag{4.1b}$$

with initial conditions

$$u(\cdot,0) = u^0, \quad z(\cdot,0) = z^0, \tag{4.2}$$

and boundary conditions

$$u_{|\partial\Omega} = 0, \tag{4.3}$$

admits a unique solution. Some a priori estimates and regularity results for the solution of (P_{uz}) are also obtained. The key tool to prove the existence and uniqueness of a solution of (P_{uz}) is to interpret this system of Partial Differential Equations (PDE) as an Ordinary Differential Equation (ODE) for the unknown function (u,z) in an appropriate Banach space.

Let us introduce some new notations. For any $r > 1$, let

$$\mathrm{V}^r(\Omega;\mathbb{R}^3) \stackrel{\text{def}}{=} \big\{u \in \mathrm{L}^2(\Omega;\mathbb{R}^3) :\ \nabla u \in \mathrm{L}^r(\Omega;\mathbb{R}^{3\times 3})\big\},$$

and for any $r \geq 2$, let

$$\mathrm{V}_0^r(\Omega;\mathbb{R}^3) \stackrel{\text{def}}{=} \big\{u \in \mathrm{V}^r(\Omega;\mathbb{R}^3) :\ u_{|\partial\Omega} = 0\big\}.$$

We endowed $\mathrm{V}^r(\Omega;\mathbb{R}^3)$ with the following norm

$$\forall u \in \mathrm{V}^r(\Omega;\mathbb{R}^3) :\ \|u\|_{\mathrm{V}^r(\Omega)} \stackrel{\text{def}}{=} \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^r(\Omega)}.$$

In the sequel, the notations for the constants introduced in the proofs are valid only in the proof.

Theorem 4.1 (Existence and uniqueness for $(\mathbf{P}_{uz})$) *Let θ be given in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$. Assume that* (3.11)*,* (3.12)*,* (3.14)*,* (3.15) *and* (3.16) *hold. Then, for any $u^0 \in \mathrm{V}_0^p(\Omega;\mathbb{R}^3)$ and for any $z^0 \in \mathrm{L}^p(\Omega;\mathcal{Z})$ the problem* (4.1)–(4.3) *possesses a unique solution $(u,z) \in \mathrm{W}^{1,q}(0,T;\mathrm{V}_0^p(\Omega;\mathbb{R}^3) \times \mathrm{L}^p(\Omega;\mathcal{Z}))$. Furthermore, the image of any bounded subset of $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ by the mapping $\theta \mapsto (u,z)$ is a bounded subset of $\mathrm{W}^{1,q}(0,T;\mathrm{V}_0^p(\Omega;\mathbb{R}^3) \times \mathrm{L}^p(\Omega;\mathcal{Z}))$.*

Proof. Let $\mathcal{F}^p(\Omega) \stackrel{\text{def}}{=} \mathrm{L}^2(\Omega;\mathbb{R}^3)\times\mathrm{L}^p(\Omega;\mathbb{R}^{3\times3}_{\mathsf{sym}})$ and $\mathcal{W}^p(\Omega) \stackrel{\text{def}}{=} \mathrm{V}_0^p(\Omega;\mathbb{R}^3)\times\mathrm{L}^p(\Omega;\mathcal{Z})$ be endowed with the norms

$$\forall\varphi \stackrel{\text{def}}{=} (\varphi_1,\varphi_2) \in \mathcal{F}^p(\Omega) : \|\varphi\|_{\mathcal{F}^p(\Omega)} \stackrel{\text{def}}{=} \|(\varphi_1,\varphi_2)\|_{\mathcal{F}^p(\Omega)} = \|\varphi_1\|_{\mathrm{L}^2(\Omega)} + \|\varphi_2\|_{\mathrm{L}^p(\Omega)},$$

and

$$\forall\psi \stackrel{\text{def}}{=} (\psi_1,\psi_2) \in \mathcal{W}^p(\Omega) : \|\psi\|_{\mathcal{W}^p(\Omega)} \stackrel{\text{def}}{=} \|(\psi_1,\psi_2)\|_{\mathcal{W}^p(\Omega)} = \|\psi_1\|_{\mathrm{V}^p(\Omega)} + \|\psi_2\|_{\mathrm{L}^p(\Omega)}.$$

It follows that $\mathcal{F}^p(\Omega)$ and $\mathcal{W}^p(\Omega)$ are two Banach spaces. We introduce now the mapping $\mathcal{A}_{\mathbb{E},\mathbb{B}_0}$ defined as follows

$$\begin{aligned} \mathcal{A}_{\mathbb{E},\mathbb{B}_0} :\ & \mathcal{W}^p(\Omega) \to \mathcal{F}^p(\Omega), \\ & (u,z) \mapsto \varphi \stackrel{\text{def}}{=} (0, \mathbb{E}(\mathrm{e}(u) - \mathbb{B}_0 z)). \end{aligned}$$

Since $\mathbb{E} \in \mathrm{L}^\infty(\Omega)$, we infer that $\mathcal{A}_{\mathbb{E},\mathbb{B}_0}$ is a linear continuous mapping from $\mathcal{W}^p(\Omega)$ to $\mathcal{F}^p(\Omega)$. Besides since $\mathbb{L}$ is a symmetric, positive definite tensor, classical results about PDE in Banach spaces imply that, for all $\varphi = (\varphi_1,\varphi_2) \in \mathcal{F}^p(\Omega)$, there exists a unique $u \in \mathrm{V}_0^p(\Omega;\mathbb{R}^3)$, denoted by $u \stackrel{\text{def}}{=} \Lambda_p(\varphi)$, such that

$$\forall v \in \mathcal{D}(\Omega) :\ \int_\Omega \mathbb{L}\mathrm{e}(u){:}\mathrm{e}(v)\,\mathrm{d}x = \int_\Omega \varphi_1{\cdot}v\,\mathrm{d}x + \int_\Omega \varphi_2{:}\mathrm{e}(v)\,\mathrm{d}x,$$

and there exists $C_{\Lambda_p} > 0$, independent of φ, such that

$$\|u\|_{\mathrm{V}^p(\Omega)} \leq C_{\Lambda_p}\big(\|\varphi_1\|_{\mathrm{L}^2(\Omega)} + \|\varphi_2\|_{\mathrm{L}^p(\Omega)}\big) = C_{\Lambda_p}\|\varphi\|_{\mathcal{F}^p(\Omega)}.$$

Hence Λ_p is linear continuous from $\mathcal{F}^p(\Omega)$ to $\mathrm{V}_0^p(\Omega;\mathbb{R}^3)$ (for more details, the reader is referred to [Val88]). It follows that (4.1a) can be rewritten as

$$\dot{u} = \Lambda_p(\varphi_{\ell,\theta}) - \Lambda_p(\mathcal{A}_{\mathbb{E},\mathbb{B}_0}(u,z)), \tag{4.4}$$

with $\varphi_{\ell,\theta} \stackrel{\text{def}}{=} (\ell, \mathbb{E}B_1 - \alpha\theta\mathrm{I})$. Next we observe that the operator $\partial\Psi + \mathbb{M}$ is strongly monotone on $\mathcal{Z}$. It follows that its inverse $(\partial\Psi + \mathbb{M})^{-1}$ is a single-valued and $\frac{1}{c^{\mathbb{M}}}$-Lipschitz continuous mapping from $\mathcal{Z}$ to $\mathcal{Z}$. We may conclude that (4.1b) can be rewritten as

$$\dot{z} = (\partial\Psi + \mathbb{M})^{-1}(\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u) - \mathbb{B}z) - \mathrm{D}_z H(z)).$$

It follows that the system (P_{uz}) is now given by the following ODE in $\mathcal{W}^p(\Omega)$

$$(\dot{u}, \dot{z}) = \mathcal{G}_p(u, z) + (\Lambda_p(\varphi_{\ell,\theta}), 0), \tag{4.5}$$

where $\mathcal{G}_p : \mathcal{W}^p(\Omega) \to \mathcal{W}^p(\Omega)$ is defined by

$$\forall (u, z) \in \mathcal{W}^p(\Omega) : \ \mathcal{G}_p(u, z) \overset{\text{def}}{=} (-\Lambda_p(\mathcal{A}_{\mathbb{E},\mathbb{B}_0}(u, z)), (\partial\Psi + \mathbb{M})^{-1}(\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u) - \mathbb{B}z) - \mathrm{D}_z H(z))).$$

Assumptions (3.14b) and (3.12b) imply that $\mathcal{G}_p$ is Lipschitz continuous on $\mathcal{W}^p(\Omega)$. Indeed, let $(u_i, z_i) \in \mathcal{W}^p(\Omega)$, $i = 1, 2$. We have

$$\begin{aligned}
&\|\mathcal{G}_p(u_1, z_1) - \mathcal{G}_p(u_2, z_2)\|_{\mathcal{W}^p(\Omega)} = \|\Lambda_p(\mathcal{A}_{\mathbb{E},\mathbb{B}_0}(u_1, z_1)) - \Lambda_p(\mathcal{A}_{\mathbb{E},\mathbb{B}_0}(u_2, z_2))\|_{\mathrm{V}^p(\Omega)} \\
&+ \|(\partial\Psi + \mathbb{M})^{-1}(\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u_1) - \mathbb{B}z_1) - (\mathrm{D}_z H(z_1)) \\
&- (\partial\Psi + \mathbb{M})^{-1}(\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u_2) - \mathbb{B}z_2) - (\mathrm{D}_z H(z_2)))\|_{\mathrm{L}^p(\Omega)} \\
&\le C_{\Lambda_p} C_{\mathcal{A}_{\mathbb{E},\mathbb{B}_0}} \big(\|u_1 - u_2\|_{\mathrm{V}^p(\Omega)} + \|z_1 - z_2\|_{\mathrm{L}^p(\Omega)}\big) + \tfrac{1}{c^{\mathbb{M}}}\|\mathrm{D}_z H(z_1) - \mathrm{D}_z H(z_2)\|_{\mathrm{L}^p(\Omega)} \\
&+ \tfrac{\|\mathbb{B}_0\|\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}}{c^{\mathbb{M}}}\big(\|\mathrm{e}(u_1) - \mathrm{e}(u_2)\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{B}_0\|\|z_1 - z_2\|_{\mathrm{L}^p(\Omega)}\big),
\end{aligned}$$

where $C_{\mathcal{A}_{\mathbb{E},\mathbb{B}_0}} \overset{\text{def}}{=} \|\mathcal{A}_{\mathbb{E},\mathbb{B}_0}\|_{\mathcal{L}(\mathcal{W}^p(\Omega), \mathcal{F}^p(\Omega))}$. By using assumption (3.12b), we infer that

$$\|\mathrm{D}_z H(z_1) - \mathrm{D}_z H(z_2)\|_{\mathrm{L}^p(\Omega)} \le C^H_{zz}\|z_1 - z_2\|_{\mathrm{L}^p(\Omega)},$$

and there exists a constant $C_{\mathcal{G}_p} > 0$, depending only on the data, such that

$$\|\mathcal{G}_p(u_1, z_1) - \mathcal{G}_p(u_2, z_2)\|_{\mathcal{W}^p(\Omega)} \le C_{\mathcal{G}_p}\big(\|u_1 - u_2\|_{\mathrm{V}^p(\Omega)} + \|z_1 - z_2\|_{\mathrm{L}^p(\Omega)}\big).$$

Therefore we can apply Cauchy-Lipschitz existence theorem for ODE (see [Car90]) and we can conclude that, for any $u^0 \in \mathrm{V}^p_0(\Omega; \mathbb{R}^3)$ and $z^0 \in \mathrm{L}^p(\Omega; \mathbb{R}^{3\times 3}_{\mathsf{sym}})$, there exists a unique solution to (P_{uz}) such that $(u, z) \in \mathrm{W}^{1,q}(0, T; \mathcal{W}^p(\Omega))$.

Let us establish now that the image of any bounded subset of $\mathrm{L}^q(0, T; \mathrm{L}^p(\Omega))$ by the mapping $\theta \mapsto (u, z)$ is a bounded subset of $\mathrm{W}^{1,q}(0, T; \mathcal{W}^p(\Omega))$. This result relies on Grönwall's lemma. Indeed, observing that $(\partial\Psi + \mathbb{M})^{-1}(0) = 0$, we get

$$\begin{aligned}
&\|(u(\cdot, t), z(\cdot, t)) - (u^0, z^0)\|_{\mathcal{W}^p(\Omega)} \le \int_0^t \|\mathcal{G}_p(u(\cdot, s), z(\cdot, s)) + (\Lambda_p(\varphi_{\ell,\theta}(\cdot, s)), 0)\|_{\mathcal{W}^p(\Omega)}\,\mathrm{d}s \\
&\le \int_0^t \|\mathcal{G}_p(u(\cdot, s), z(\cdot, s)) - \mathcal{G}_p(u^0, z^0)\|_{\mathcal{W}^p(\Omega)}\,\mathrm{d}s + \int_0^t \|\mathcal{G}_p(u^0, z^0) + (\Lambda_p(\varphi_{\ell,\theta}(\cdot, s)), 0)\|_{\mathcal{W}^p(\Omega)}\,\mathrm{d}s \\
&\le C_{\mathcal{G}_p}\int_0^t \big(\|u(\cdot, s) - u^0\|_{\mathrm{V}^p(\Omega)} + \|z(\cdot, s) - z^0\|_{\mathrm{L}^p(\Omega)}\big)\,\mathrm{d}s + t\big(\|\mathcal{G}_p(u^0, z^0)\|_{\mathcal{W}^p(\Omega)} \\
&+ C_{\Lambda_p}\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}\|B_1\||\Omega|^{\frac{1}{p}}\big) + C_{\Lambda_p}\int_0^t \big(\|\ell(\cdot, s)\|_{\mathrm{L}^2(\Omega)} + \sqrt{3}\alpha\|\theta(\cdot, s)\|_{\mathrm{L}^p(\Omega)}\big)\,\mathrm{d}s,
\end{aligned}$$

for all $t \in [0, T]$. Hence, we find

$$\begin{aligned}
&\|(u(\cdot, t), z(\cdot, t)) - (u^0, z^0)\|_{\mathcal{W}^p(\Omega)} \le \exp(C_{\mathcal{G}_p} t)\Big(t\big(\|\mathcal{G}_p(u^0, z^0)\|_{\mathcal{W}^p(\Omega)} \\
&+ C_{\Lambda_p}\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}\|B_1\||\Omega|^{\frac{1}{p}}\big) + \int_0^t C_{\Lambda_p}\big(\|\ell(\cdot, s)\|_{\mathrm{L}^2(\Omega)} + \sqrt{3}\alpha\|\theta(\cdot, s)\|_{\mathrm{L}^p(\Omega)}\big)\,\mathrm{d}s\Big).
\end{aligned}$$

Now we go back to (4.5). We observe that

$$\begin{aligned}
&\|\big(\dot u(\cdot,t),\dot z(\cdot,t)\big)\|_{\mathcal{W}^p(\Omega)} = \|\mathcal{G}_p(u(\cdot,t),z(\cdot,t))+(\Lambda_p(\varphi_{\ell,\theta}(\cdot,t)),0)\|_{\mathcal{W}^p(\Omega)}\\
&\le C_{\mathcal{G}_p}\big(\|u(\cdot,t)-u^0\|_{\mathrm{V}^p(\Omega)}+\|z(\cdot,t)-z^0\|_{\mathrm{L}^p(\Omega)}\big)+\|\mathcal{G}_p(u^0,z^0)\|_{\mathcal{W}^p(\Omega)}\\
&+C_{\Lambda_p}\big(\|\ell(\cdot,t)\|_{\mathrm{L}^2(\Omega)}+\sqrt{3}\alpha\|\theta(\cdot,t)\|_{\mathrm{L}^p(\Omega)}+\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}\|B_1\||\Omega|^{\frac{1}{p}}\big),
\end{aligned}$$

for almost every $t\in[0,T]$. Besides there exists a generic constant $C>0$ such that

$$\|(\dot u,\dot z)\|_{\mathrm{L}^q(0,T;\mathcal{W}^p(\Omega))}\le C\big(\|\mathcal{G}_p(u^0,z^0)\|_{\mathcal{W}^p(\Omega)}+\|\ell\|_{\mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))}+\alpha\|\theta\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))}+1\big).$$

This concludes the proof. $\square$

We establish that the mapping $\widetilde{\vartheta}\mapsto(u,z)$, where (u,z) is the unique solution of (P_{uz}) when $\theta=\zeta(\widetilde{\vartheta})$, is continuous from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{W}^{1,\bar q}(0,T;\mathrm{V}_0^{\bar p}(\Omega)\times\mathrm{L}^{\bar p}(\Omega))$.

Lemma 4.2 *Assume that* (3.11)*,* (3.12)*,* (3.14)*,* (3.15) *and* (3.16) *hold and* $u^0\in\mathrm{V}_0^p(\Omega;\mathbb{R}^3)$ *and* $z^0\in\mathrm{L}^p(\Omega;\mathcal{Z})$*. Then* $\vartheta\mapsto(u,z)$ *is continuous from* $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ *into* $\mathrm{W}^{1,\bar q}(0,T;\mathrm{V}_0^{\bar p}(\Omega;\mathbb{R}^3)\times\mathrm{L}^{\bar p}(\Omega;\mathcal{Z}))$.

Proof. Let $\widetilde{\vartheta}_i\in\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ and for $i=1,2$, we denote by $\theta_i\overset{\text{def}}{=}\zeta(\widetilde{\vartheta}_i)\in\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ and (u_i,z_i) the solution of the following system:

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u_i)-\mathbb{B}z_i)+\alpha\theta_i\mathrm{I}+\mathbb{L}\mathrm{e}(\dot u_i))=\ell, \tag{4.6a}$$

$$\partial\Psi(\dot z_i)+\mathbb{M}\dot z_i-\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u_i)-\mathbb{B}z_i)+\mathrm{D}_zH(z_i)\ni 0, \tag{4.6b}$$

together with initial conditions

$$u_i(\cdot,0)=u^0,\quad z_i(\cdot,0)=z^0, \tag{4.7}$$

and boundary conditions

$$u_{i|\partial\Omega}=0. \tag{4.8}$$

Since the mapping $\phi^{\widetilde{\vartheta},\theta}:\widetilde{\vartheta}\mapsto\theta=\zeta(\widetilde{\vartheta})$ is continuous from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$, we only need to check that the mapping $\theta=\zeta(\widetilde{\vartheta})\mapsto(u,z)$ is continuous from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{W}^{1,\bar q}(0,T;\mathcal{W}^{\bar p}(\Omega))$ where the notation $\mathcal{W}^{\bar p}(\Omega)$ was introduced in the proof of Theorem 4.1.

With the same kind of computations as previously, we get

$$\begin{aligned}
&\|(u_1(\cdot,t),z_1(\cdot,t))-(u_2(\cdot,t),z_2(\cdot,t))\|_{\mathcal{W}^p(\Omega)}\le\int_0^t\|\mathcal{G}_p(u_1(\cdot,s),z_1(\cdot,s))\\
&-\mathcal{G}_p(u_2(\cdot,s),z_2(\cdot,s))+(\Lambda_p(\varphi_{\ell,\theta_1}(\cdot,s)),0)-(\Lambda_p(\varphi_{\ell,\theta_2}(\cdot,s)),0)\|_{\mathcal{W}^p(\Omega)}\,\mathrm{d}s\\
&\le C_{\mathcal{G}_p}\int_0^t\big(\|u_1(\cdot,s)-u_2(\cdot,s)\|_{\mathrm{V}^p(\Omega)}+\|z_1(\cdot,s)-z_2(\cdot,s)\|_{\mathrm{L}^p(\Omega)}\big)\,\mathrm{d}s\\
&+\sqrt{3}C_{\Lambda_p}\alpha\int_0^t\|\theta_1(\cdot,s)-\theta_2(\cdot,s)\|_{\mathrm{L}^p(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

for all $t\in[0,T]$. We may consider the case where $p=\bar p=2$ and, by applying Grönwall's lemma, we obtain

$$\|(u_1(\cdot,t)-u_2(\cdot,t),z_1(\cdot,t)-z_2(\cdot,t))\|_{\mathcal{W}^{\bar p}(\Omega)}\le\sqrt{3}C_{\Lambda_{\bar p}}\alpha\exp(C_{\mathcal{G}_{\bar p}}t)t^{\frac{\bar q-1}{\bar q}}\|\theta_1-\theta_2\|_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))},$$

for all $t \in [0,T]$. Then we infer that

$$\begin{aligned}&\|(\dot u_1(\cdot,t)-\dot u_2(\cdot,t),\dot z_1(\cdot,t)-\dot z_2(\cdot,t))\|_{\mathcal{W}^{\bar p}(\Omega)} = \|\mathcal{G}_{\bar p}(u_1(\cdot,t),z_1(\cdot,t))\\ &-\mathcal{G}_{\bar p}(u_2(\cdot,t),z_2(\cdot,t))+(\Lambda_{\bar p}(\varphi_{\ell,\theta_1}(\cdot,t)),0)-(\Lambda_{\bar p}(\varphi_{\ell,\theta_2}(\cdot,t)),0)\|_{\mathcal{W}^{\bar p}(\Omega)}\\ &\le C_{\mathcal{G}_{\bar p}}\big(\|u_1(\cdot,t)-u_2(\cdot,t)\|_{\mathrm{V}^{\bar p}(\Omega)}+\|z_1(\cdot,t)-z_2(\cdot,t)\|_{\mathrm{L}^{\bar p}(\Omega)}\big)\\ &+\sqrt{3}C_{\Lambda_{\bar p}}\alpha\|\theta_1(\cdot,t)-\theta_2(\cdot,t)\|_{\mathrm{L}^{\bar p}(\Omega)},\end{aligned}$$

for almost every $t \in [0,T]$. Hence, with $q=\bar q$, we get

$$\|(\dot u_1-\dot u_2,\dot z_1-\dot z_2)\|_{\mathrm{L}^{\bar q}(0,T;\mathcal{W}^{\bar p}(\Omega))} \le \sqrt{3}C_{\Lambda_{\bar p}}\alpha\big(C_{\mathcal{G}_{\bar p}}\exp(C_{\mathcal{G}_{\bar p}}T)\big(\tfrac{T^{\bar q}}{\bar q}\big)^{\frac{1}{\bar q}}+1\big)\|\theta_1-\theta_2\|_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))},$$

which allows us to conclude. □

5 Existence, uniqueness and regularity results for the enthalpy equation

We recall in this section existence, uniqueness and some regularity results for the enthalpy equation. More precisely, let us assume that $\vartheta^0 \in \mathrm{L}^2(\Omega)$ and $f \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')$. Furthermore, let $\widetilde\kappa^c \in \mathrm{L}^\infty(\mathcal{Q}_T;\mathbb{R}^{3\times3}_{\mathsf{sym}})$ be such that

$$\exists c^{\kappa^c}>0\,\forall v\in\mathbb{R}^3:\ \widetilde\kappa^c(x,t)v\cdot v \ge c^{\kappa^c}|v|^2 \ \text{ a.e. } (x,t)\in\mathcal{Q}_T, \tag{5.1a}$$

$$\exists C^{\kappa^c}>0:\ |\widetilde\kappa^c(x,t)| \le C^{\kappa^c} \ \text{ a.e. } (x,t)\in\mathcal{Q}_T. \tag{5.1b}$$

We consider the following problem (P_ϑ)

$$\dot\vartheta - \mathrm{div}(\widetilde\kappa^c\nabla\vartheta) = f, \tag{5.2}$$

with initial conditions

$$\vartheta(0)=\vartheta^0, \tag{5.3}$$

and boundary conditions

$$\widetilde\kappa^c\nabla\vartheta\cdot\eta_{|\partial\Omega}=0. \tag{5.4}$$

The weak formulation is given by

$$\begin{cases}\text{Find } \vartheta:[0,T]\to\mathrm{H}^1(\Omega) \text{ such that } \vartheta(0)=\vartheta^0 \text{ and for all } \xi\in\mathrm{H}^1(\Omega),\\ \displaystyle\int_\Omega \dot\vartheta\xi\,\mathrm{d}x+\int_\Omega\widetilde\kappa^c\nabla\vartheta\cdot\nabla\xi\,\mathrm{d}x=\int_\Omega f\xi\,\mathrm{d}x \ \text{ in the sense of distributions.}\end{cases} \tag{5.5}$$

Theorem 5.1 (Existence and uniqueness for $(\mathbf{P}_\vartheta)$) *Assume that the previous assumptions hold, then the problem* (5.2)–(5.4) *possesses a unique solution* $\vartheta \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))\cap\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ *with* $\dot\vartheta\in\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))$. *Moreover we have*

$$\|\vartheta(\tau)\|^2_{\mathrm{L}^2(\Omega)}+2c^{\kappa^c}\int_0^\tau\|\nabla\vartheta(t)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t \le \exp(c^{\kappa^c}T)\big(\|\vartheta^0\|^2_{\mathrm{L}^2(\Omega)}+\tfrac{1}{c^{\kappa^c}}\|f\|^2_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')}\big),$$

for all $\tau\in[0,T]$.

Proof. The proof of existence and uniqueness of a solution is quite classical and can be found in [Lio68]. The estimate is straightforward and its verification is left to the reader. $\square$

Let us introduce the following functional space

$$\mathrm{W}^{\vartheta} \overset{\text{def}}{=} \big\{\vartheta \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) : \ \dot{\vartheta} \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))\big\},$$

endowed with the norm

$$\forall \vartheta \in \mathrm{W}^{\vartheta} : \ \|\vartheta\|_{\mathrm{W}^{\vartheta}} \overset{\text{def}}{=} \|\vartheta\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\vartheta\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} + \|\dot{\vartheta}\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)')}.$$

Due to [Sim87], we know that W^{ϑ} is compactly embedded in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$. Note that the previous estimate implies that there exists a generic constant $C > 0$ such that the solution of problem (P_{ϑ}) satisfies

$$\|\vartheta\|_{\mathrm{W}^{\vartheta}} \leq C\big(\|\vartheta^0\|_{\mathrm{L}^2(\Omega)} + \|f\|_{\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))')}\big).$$

6 Local existence result

We provide in this section a local existence result for (3.7)–(3.9) by using a fixed-point argument. To this aim, for any given $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we define $\widetilde{\kappa}^c \overset{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \theta)$ and $f = f^{\widetilde{\vartheta}} \overset{\text{def}}{=} \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})) + \Psi(\dot{z}) + \mathbb{M}\dot{z}.\dot{z}$ in (P_{ϑ}), where (u,z) are the solutions of (P_{uz}) with $\theta = \zeta(\widetilde{\vartheta})$. We assume that $\vartheta^0 \in \mathrm{L}^2(\Omega)$, $u^0 \in \mathrm{V}^p_0(\Omega;\mathbb{R}^3)$ and $z^0 \in \mathrm{L}^p(\Omega;\mathcal{Z})$ with $p \in [4, \min(\beta_1\bar{p}, 6)]$ and we choose $q = \beta_1\bar{q}$. With the results obtained in Section 4, we infer that $f^{\widetilde{\vartheta}}$ belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and we can define $\vartheta \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{W}^{\vartheta}$ as the unique solution of (P_{ϑ}). Thus we can introduce the fixed point mapping $\phi^{\widetilde{\vartheta},\vartheta} : \widetilde{\vartheta} \mapsto \vartheta$ from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$.

Proposition 6.1 *The mapping $\phi^{\widetilde{\vartheta},\vartheta}$ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$.*

Proof. This proof is quite similar to the proof of Proposition 6.1 in [PaP11]. Let $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ be a converging sequence of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ and let $\widetilde{\vartheta}_*$ be its limit. We denote by $\vartheta_n \overset{\text{def}}{=} \phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_n)$ for all $n \geq 0$ and $\vartheta_* = \phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_*)$. Since $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ is a bounded family of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, the previous results imply that $(\vartheta_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{W}^{\vartheta}$. Then it follows that $(\vartheta_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ (see [Sim87]) and it is possible to extract a subsequence, still denoted by $(\vartheta_n)_{n\in\mathbb{N}}$, such that

$$\begin{aligned}
&\vartheta_n \rightharpoonup \vartheta \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \ \text{ weak},\\
&\dot{\vartheta}_n \rightharpoonup \dot{\vartheta} \ \text{ in } \ \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))') \ \text{ weak},\\
&\vartheta_n \to \vartheta \ \text{ in } \ \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega)).
\end{aligned}$$

Let us define $\mathcal{V}_T \overset{\text{def}}{=} \big\{w \in \mathrm{C}^\infty([0,T]) : \ w(T) = 0\big\}$. Hence we observe that for all $n \geq 0$, we have

$$\begin{aligned}
&\forall \xi \in \mathrm{H}^1(\Omega)\ \forall w \in \mathcal{V}_T :\\
&-\int_{\mathcal{Q}_T} \vartheta_n(x,t)\xi(x)\dot{w}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_T} \widetilde{\kappa}^c_n \nabla\vartheta_n(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} \vartheta^0(x)\xi(x)w(0)\,\mathrm{d}x,
\end{aligned} \tag{6.1}$$

with $\widetilde{\kappa}_n^c \overset{\text{def}}{=} \kappa^c\big(\mathrm{e}(u_n), z_n, \theta_n\big)$ and (u_n, z_n) solutions of (P_{uz}) with $\theta_n \overset{\text{def}}{=} \zeta(\widetilde{\vartheta}_n)$. Since $(\widetilde{\vartheta}_n)_{n\in\mathbb{N}}$ converges to $\widetilde{\vartheta}_*$ in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we infer from Lemma 4.2 that $(u_n, z_n)_{n\in\mathbb{N}}$ converges to the solution (u_*, z_*) of (P_{uz}) with $\theta_* = \zeta(\widetilde{\vartheta}_*)$ in $\mathrm{W}^{1,\bar{q}}(0,T;\mathrm{V}_0^{\bar{p}}(\Omega)\times\mathrm{L}^{\bar{p}}(\Omega))$. Furthermore the mapping $\phi^{\widetilde{\vartheta},\theta} : \widetilde{\vartheta} \mapsto \theta = \zeta(\widetilde{\vartheta})$ is Lipschitz continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, which implies that it is possible to extract subsequences, still denoted by θ_n, u_n, z_n, such that

$$\theta_n, \mathrm{e}(u_n), z_n \to \theta_*, \mathrm{e}(u_*), z_* \text{ a.e. } (x,t) \in \mathcal{Q}_T.$$

Note that the continuity of the mapping κ^c gives

$$\widetilde{\kappa}_n^c = \kappa^c(\mathrm{e}(u_n), z_n, \theta_n) \to \widetilde{\kappa}_*^c = \kappa^c(\mathrm{e}(u_*), z_*, \theta_*) \text{ a.e. } (x,t) \in \mathcal{Q}_T,$$

and due to the boundedness assumption on κ^c, with the help of Lebesgue's theorem, we obtain

$$\widetilde{\kappa}_n^c \nabla\xi w \to \widetilde{\kappa}_*^c \nabla\xi w \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)).$$

Therefore it is possible to pass to the limit in all the terms of the left hand side of (6.1) to get

$$\begin{aligned}\forall \xi \in \mathrm{H}^1(\Omega)\ \forall w \in \mathcal{V}_T :\ & -\int_{\mathcal{Q}_T} \vartheta(x,t)\xi(x)\dot{w}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_T} \widetilde{\kappa}_*^c \nabla\vartheta(x,t)\nabla\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{n\to+\infty}\int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \vartheta^0(x)\xi(x)w(0)\,\mathrm{d}x.\end{aligned}$$

Since $p \geq 4$, we may deduce that the mapping $\widetilde{\vartheta} \mapsto f^{\widetilde{\vartheta}}$ is continuous from $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{r_1}(0,T;\mathrm{L}^{r_2}(\Omega))$ with $\frac{1}{r_1} = \frac{1}{\beta_1\bar{q}} + \frac{1}{\bar{q}} \leq \frac{3}{2\bar{q}} < \frac{3}{4}$ and $\frac{1}{r_2} = \frac{1}{p} + \frac{1}{\bar{p}} \leq \frac{3}{4}$. Indeed, for any $\widetilde{\vartheta}_i$ in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, let (u_i, z_i) be the solution of (P_{uz}) with $\theta_i = \zeta(\widetilde{\vartheta}_i)$, $i = 1, 2$. We have

$$\begin{aligned}f^{\widetilde{\vartheta}_1} - f^{\widetilde{\vartheta}_2} &= \mathbb{L}\mathrm{e}(\dot{u}_1+\dot{u}_2):\mathrm{e}(\dot{u}_1-\dot{u}_2) + \alpha(\theta_1-\theta_2)\mathrm{tr}(\mathrm{e}(\dot{u}_1)) \\ &+ \alpha\theta_2\mathrm{tr}(\mathrm{e}(\dot{u}_1-\dot{u}_2)) + \Psi(\dot{z}_1) - \Psi(\dot{z}_2) + \mathbb{M}(\dot{z}_1+\dot{z}_2).(\dot{z}_1-\dot{z}_2).\end{aligned}$$

On the other hand, (3.11c) and (3.11b) imply

$$|\Psi(\dot{z}_1)-\Psi(\dot{z}_2)| \leq C^\Psi|\dot{z}_1-\dot{z}_2|.$$

The boundedness and the continuity properties proved in Theorem 4.1 and Lemma 4.2, respectively, allow us to deduce the desired result. Therefore, reminding that $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^6(\Omega) = (\mathrm{L}^{\frac{6}{5}}(\Omega))'$ with continuous embedding, we may infer that

$$\forall \xi \in \mathrm{H}^1(\Omega)\ \forall w \in \mathcal{V}_T :\ \lim_{n\to+\infty}\int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_n}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_T} f^{\widetilde{\vartheta}_*}(x,t)\xi(x)w(t)\,\mathrm{d}x\,\mathrm{d}t.$$

We conclude that ϑ is solution of problem (P_ϑ) with the data $\widetilde{\kappa}_*^c$ and $f^{\widetilde{\vartheta}_*}$. Moreover by uniqueness of the solution, it follows that $\vartheta = \vartheta_*$ and the whole sequence $(\vartheta_n)_{n\in\mathbb{N}}$ converges to $\vartheta_* = \zeta(\widetilde{\vartheta}_*)$. $\square$

We provide below that the mapping $\phi^{\widetilde{\vartheta},\vartheta}$ fulfills the other assumptions of Schauder's fixed point theorem. We begin by introducing some notations; let $R^0, R^\vartheta > 0$ be any given positive real numbers such that

$$\max\big(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{L}^p(\Omega)}\big) \leq R^0 \quad\text{and}\quad \|\widetilde{\vartheta}\|_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))} \leq R^\vartheta.$$

Since $\theta = \zeta(\widetilde{\vartheta})$, it is clear that

$$\|\theta\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))} \leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta_1}} |\Omega|^{\frac{\beta_1\bar{p}-p}{\beta_1\bar{p}p}} \|\widetilde{\vartheta}\|^{\frac{1}{\beta_1}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))} \leq R^\theta \overset{\text{def}}{=} \big(\tfrac{\beta_1}{c^c} R^\vartheta\big)^{\frac{1}{\beta_1}} |\Omega|^{\frac{\beta_1\bar{p}-p}{\beta_1\bar{p}p}},$$

with $q = \beta_1\bar{q} > 4$. Thanks to results of Section 4, we may infer that there exists $R^f \overset{\text{def}}{=} R^f(R^0, R^\theta, \|\ell\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}) > 0$, depending on R^0, R^θ and $\|\ell\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}$, such that

$$\|f^{\widetilde{\vartheta}}\|_{\mathrm{L}^{q/2}(0,T;\mathrm{L}^{p/2}(\Omega))} \leq R^f.$$

The results of Section 5 imply that there exists a generic constant $C > 0$ such that

$$\begin{aligned}
&\|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \leq C(\|f^{\widetilde{\vartheta}}\|_{\mathrm{L}^2(0,\tau;\mathrm{L}^2(\Omega))} + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}) \\
&\leq C\big(|\Omega|^{\frac{p-4}{2p}} T^{\frac{q-4}{2q}} \|f^{\widetilde{\vartheta}}\|_{\mathrm{L}^{q/2}(0,\tau;\mathrm{L}^{p/2}(\Omega))} + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big) \leq C\big(|\Omega|^{\frac{p-4}{2p}} T^{\frac{q-4}{2q}} R^f + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big).
\end{aligned}$$

Now let $0 < \tau \leq T$ and $\mathrm{W}^\vartheta_\tau$ be the following functional space

$$\mathrm{W}^\vartheta_\tau \overset{\text{def}}{=} \big\{\vartheta \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega)) :\ \dot{\vartheta} \in \mathrm{L}^2(0,\tau;(\mathrm{H}^1(\Omega))')\big\}.$$

For any $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega)))$, we define its extension $\widetilde{\vartheta}_{\text{ext}}$ as follows

$$\widetilde{\vartheta}_{\text{ext}}(x,t) \overset{\text{def}}{=} \begin{cases} \widetilde{\vartheta}(x,t) & \text{if } (x,t) \in \Omega \times [0,\tau], \\ 0 & \text{if } (x,t) \in \Omega \times (\tau, T]. \end{cases}$$

Therefore it is plain that $\widetilde{\vartheta}_{\text{ext}} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$. Furthermore the mapping $\widetilde{\vartheta} \mapsto \widetilde{\vartheta}_{\text{ext}}$ is a contraction from $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$ into $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$. For any $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$, we define $\phi^{\widetilde{\vartheta},\vartheta}_\tau(\widetilde{\vartheta})$ as the restriction on $[0,\tau]$ of $\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})$. It follows from Proposition 6.1 that $\phi^{\widetilde{\vartheta},\vartheta}_\tau$ is continuous from $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$. Recalling that $\bar{p} = 2$, for any $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$, we have

$$\begin{aligned}
&\|\phi^{\widetilde{\vartheta},\vartheta}_\tau(\widetilde{\vartheta})\|_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))} = \|\phi^{\widetilde{\vartheta},\vartheta}_\tau(\widetilde{\vartheta})\|_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))} \\
&= \Big(\int_0^\tau \|\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}}(\cdot,t))\|^{\bar{q}}_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac{1}{\bar{q}}} \leq \tau^{\frac{1}{\bar{q}}} \|\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))},
\end{aligned}$$

and the previous estimates allow us to show that, for any $R^\vartheta > 0$, there exists $\tau \in (0,T]$ such that $\phi^{\widetilde{\vartheta},\vartheta}_\tau$ maps the closed ball $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ into itself. Note that the image of $\bar{B}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ by $\phi^{\widetilde{\vartheta},\vartheta}$ is a bounded subset of W^ϑ and thus it is relatively compact in $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$. It follows that the image of $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))}(0,R^\vartheta)$ by $\phi^{\widetilde{\vartheta},\vartheta}_\tau$ is also relatively compact in $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$. Finally we conclude that the problem (3.7)–(3.9) possesses a local solution (u,z,ϑ) defined on $[0,\tau]$ such that $(u,z) \in \mathrm{W}^{1,q}(0,\tau;\mathrm{V}^p_0(\Omega) \times \mathrm{L}^p(\Omega))$ and $\vartheta \in \mathrm{W}^\vartheta_\tau$.

We have to go back to the problem (3.2)–(3.4). First we observe that g and ζ define a C^1-diffeomorphism from $(0,\infty)$ to $(0,\infty)$ and any solution of (3.7)–(3.9) provides a solution of (3.2)–(3.4) as soon as the enthalpy ϑ remains strictly positive. So we assume now that the initial enthalpy is strictly positive almost everywhere on Ω, i.e., there exists $\bar{\vartheta} > 0$ such that

$$g(\theta^0) = \vartheta^0 \in \mathrm{L}^2(\Omega), \quad g(\theta^0(x)) = \vartheta^0(x) \geq \bar{\vartheta} > 0 \text{ a.e. } x \in \Omega. \tag{6.2}$$

Therefore the Stampacchia's truncation method allows us to provide a local existence result for the problem (3.2)–(3.4).

Theorem 6.2 (Local existence result) *Assume that* (3.11), (3.12), (3.14), (3.15), (3.16), (3.17) *and* (6.2) *hold. Then, for any* $u^0 \in \mathrm{V}_0^p(\Omega;\mathbb{R}^3)$ *and* $z^0 \in \mathrm{L}^p(\Omega;\mathcal{Z})$ *with* $p \in [4, \min(\beta_1\bar{p}, 6)]$*, there exists* $\tau \in (0,T]$ *such that the problem* (3.2)–(3.4) *admits a solution on* $[0,\tau]$.

Proof. Let (u,z,ϑ) be a solution of (3.7)–(3.9) on $\big[0,\tau\big]$. We prove now that

$$\vartheta(x,t) > 0 \ \text{ a.e. } \ (x,t) \in \mathcal{Q}_\tau.$$

Let us define $\varphi : [0,\tau] \to \mathbb{R}$ such that

$$\forall t \in [0,\tau] : \ \varphi(t) \overset{\text{def}}{=} \bar{\vartheta} \exp\big(-\tfrac{(3\alpha)^2\beta_1}{2c^{\mathbb{L}}c^c} t\big), \tag{6.3}$$

and let $G \in \mathrm{C}^1(\mathbb{R})$ be such that

(i) $\exists C^G > 0 \, \forall \sigma \in \mathbb{R} : \ |G'(\sigma)| \le C^G$,

(ii) G is strictly increasing on $(0,\infty)$,

(iii) $\forall \sigma \le 0 : \ G(\sigma) = 0$.

Moreover let $\Gamma(\sigma) \overset{\text{def}}{=} \int_0^\sigma G(s)\,\mathrm{d}s$ for all $\sigma \in \mathbb{R}$, $\vartheta_\varphi \overset{\text{def}}{=} -\vartheta + \varphi$ and $\Xi(t) \overset{\text{def}}{=} \int_\Omega \Gamma(\vartheta_\varphi)\,\mathrm{d}x$. It is plain that $\Gamma \in \mathrm{C}^2(\mathbb{R};\mathbb{R})$ and $\Gamma(\sigma) > 0$ for all $\sigma > 0$. Since $\vartheta_\varphi(0) = -\vartheta^0 + \bar{\vartheta} \le 0$ almost everywhere on Ω, we may deduce that $\Xi(0) = 0$. Observe that $\vartheta \in \mathrm{W}_\tau^\vartheta$ and $\varphi \in \mathrm{C}^\infty([0,\tau];\mathbb{R})$, it follows that Ξ is absolutely continuous and

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\Xi(t) = \int_\Omega G(\vartheta_\varphi)\dot{\vartheta}_\varphi\,\mathrm{d}x \\
&= -\int_\Omega G(\vartheta_\varphi)(\mathrm{div}(\widetilde{\kappa}^c\nabla\vartheta){+}\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})){+}\Psi(\dot{z}){+}\mathbb{M}\dot{z}.\dot{z}{-}\dot{\varphi})\,\mathrm{d}x \\
&= -\int_\Omega G'(\vartheta_\varphi)\widetilde{\kappa}^c\nabla\vartheta_\varphi{:}\nabla\vartheta_\varphi\,\mathrm{d}x - \int_\Omega G(\vartheta_\varphi)(\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})){+}\Psi(\dot{z}){+}\mathbb{M}\dot{z}.\dot{z}{-}\dot{\varphi})\,\mathrm{d}x,
\end{aligned}$$

for almost every $t \in [0,\tau]$. Using (3.15) and Cauchy-Schwarz's inequality, we get

$$\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})) \ge c^{\mathbb{L}}|\mathrm{e}(\dot{u})|^2 - 3\alpha|\theta||\mathrm{e}(\dot{u})| \ge \tfrac{c^{\mathbb{L}}}{2}|\mathrm{e}(\dot{u})|^2 - \tfrac{(3\alpha)^2|\theta|^2}{2c^{\mathbb{L}}}.$$

But $G'(\vartheta_\varphi) \ge 0$ and $G(\vartheta_\varphi) \ge 0$ almost everywhere and (3.11b) and (3.17d) hold, so we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Xi(t) \le \int_\Omega G(\vartheta_\varphi)\big(\tfrac{(3\alpha)^2|\theta|^2}{2c^{\mathbb{L}}}{+}\dot{\varphi}\big)\,\mathrm{d}x \ \text{ a.e. } \ t \in [0,\tau].$$

Furthermore $\theta = \zeta(\vartheta)$ and $\beta_1 \ge 2$, thus with (3.18), we obtain

$$|\theta| = |\zeta(\vartheta)| \le \sqrt{\tfrac{\beta_1}{c^c}\max(\vartheta,0){+}1} - 1 \le \sqrt{\tfrac{\beta_1}{c^c}\max(\vartheta,0)} \ \text{ a.e. } \ (x,t) \in \mathcal{Q}_\tau.$$

Finally, observing that $G(\vartheta_\varphi)$ vanishes whenever $\vartheta \ge \varphi$, we infer that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Xi(t) \le \int_\Omega G(\vartheta_\varphi)\big(\tfrac{(3\alpha)^2\beta_1}{2c^{\mathbb{L}}c^c}\varphi{+}\dot{\varphi}\big)\,\mathrm{d}x = 0 \ \text{ a.e. } \ t \in [0,\tau].$$

We may deduce that $\Xi(t) \le \Xi(0) = 0$ for all $t \in [0,\tau]$, and thus we have

$$\Gamma(\vartheta_\varphi) \le 0 \ \text{ a.e. } \ (x,t) \in \Omega \times (0,\tau),$$

which, implies that

$$\vartheta_\varphi = -\vartheta + \varphi \le 0 \ \text{ a.e. } \ (x,t) \in \Omega \times (0,\tau).$$

□

7 Global existence result

Let us begin this section with some a priori estimates for the solutions of the problem (3.7)–(3.9). The result relies on an energy balance combined with Grönwall's lemma and on more specific techniques for the heat equation introduced by Boccardo and Gallouët in [BoG89, BoG92]. Then, by a contradiction argument, we will prove that the problem (3.2)–(3.4) possesses a global solution (u, z, θ).

Proposition 7.1 (Global energy estimate) *Assume that* (3.11), (3.12), (3.14), (3.15), (3.16) *and* (3.17) *hold. Assume moreover that* $u^0 \in \mathrm{V}_0^p(\Omega;\mathbb{R}^3)$, $z^0 \in \mathrm{L}^p(\Omega;\mathcal{Z})$ *with* $p \in [4, \min(\beta_1\bar{p}, 6)]$ *and* $\vartheta^0 \in \mathrm{L}^2(\Omega)$ *such that* (6.2) *holds. Then, there exists a constant* $\widetilde{C} > 0$*, depending only on* $\|u^0\|_{\mathrm{H}^1(\Omega)}$, $\|z^0\|_{\mathrm{H}^1(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}$ *and the data such that for any solution* (u, z, ϑ) *of problem* (3.7)–(3.9) *defined on* $[0,\tau]$, $\tau \in (0, T]$*, we have*

$$\|\dot{u}\|^2_{\mathrm{L}^2(0,\widetilde{\tau};\mathrm{H}^1(\Omega))} + \|\dot{z}\|^2_{\mathrm{L}^2(0,\widetilde{\tau};\mathrm{L}^2(\Omega))} + \|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} + \|\vartheta(\cdot,\widetilde{\tau})\|_{\mathrm{L}^1(\Omega)} \leq \widetilde{C}, \tag{7.1}$$

for all $\widetilde{\tau} \in [0,\tau]$*. Furthermore, for any* $r \in \left[1, \frac{5}{4}\right)$ *there exists a constant* $C_r > 0$*, depending only on* r *and* $\widetilde{C}$ *such that*

$$\|\nabla\vartheta\|_{\mathrm{L}^r(0,\tau;\mathrm{L}^r(\Omega))} \leq C_r. \tag{7.2}$$

Proof. On the one hand, we multiply (3.7a) by $\dot{u}$ and we integrate this expression over $\mathcal{Q}_{\widetilde{\tau}}$, with $\widetilde{\tau} \in [0,\tau]$, to get

$$\int_{\mathcal{Q}_{\widetilde{\tau}}} (\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z)+\alpha\theta\mathrm{I}+\mathbb{L}\mathrm{e}(\dot{u})):\mathrm{e}(\dot{u})\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_{\widetilde{\tau}}} \ell\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}t. \tag{7.3}$$

On the other hand, by using the definition of the subdifferential $\partial\Psi(\dot{z})$ and (3.11a), we deduce from (3.7b) that

$$\int_{\mathcal{Q}_{\widetilde{\tau}}} (\mathbb{M}\dot{z}-\mathbb{B}_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u)-\mathbb{B}z)+\mathrm{D}_z H(z)).\dot{z}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_{\widetilde{\tau}}} \Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{7.4}$$

Adding (7.3) and (7.4), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \mathbb{E}(\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau})):(\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau}))\,\mathrm{d}x + \int_{\mathcal{Q}_{\widetilde{\tau}}} \mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathcal{Q}_{\widetilde{\tau}}} \mathbb{M}\dot{z}.\dot{z}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega H(z(\cdot,\widetilde{\tau}))\,\mathrm{d}x + \int_{\mathcal{Q}_{\widetilde{\tau}}} \alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u}))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathcal{Q}_{\widetilde{\tau}}} \Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}t = C_0^{u,z} + \int_{\mathcal{Q}_{\widetilde{\tau}}} \ell\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{7.5}$$

where $C_0^{u,z} \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega \mathbb{E}(\mathrm{e}(u^0)-\mathbb{B}z^0):(\mathrm{e}(u^0)-\mathbb{B}z^0)\,\mathrm{d}x + \int_\Omega H(z^0)\,\mathrm{d}x$. Now we integrate (3.7c) over $\mathcal{Q}_{\widetilde{\tau}}$. By taking into account the boundary conditions (3.8), we get

$$\begin{aligned}
\int_\Omega \vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x &= \int_\Omega \vartheta(\cdot,0)\,\mathrm{d}x + \int_{\mathcal{Q}_{\widetilde{\tau}}} \mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathcal{Q}_{\widetilde{\tau}}} \mathbb{M}\dot{z}.\dot{z}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_{\widetilde{\tau}}} \alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u}))\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_{\widetilde{\tau}}} \Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We multiply this last equality by $\frac{1}{2}$ and we add it to (7.5). Thanks to (3.12a), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \mathbb{E}(\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau})):(\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau}))\,\mathrm{d}x + c^H\|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)}\\
&+\tfrac{1}{2}\int_{\mathcal{Q}_{\widetilde{\tau}}} \mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\int_{\mathcal{Q}_{\widetilde{\tau}}}\mathbb{M}\dot{z}.\dot{z}\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\int_{\mathcal{Q}_{\widetilde{\tau}}}\Psi(\dot{z})\,\mathrm{d}x\,\mathrm{d}t\\
&+\tfrac{1}{2}\int_\Omega \vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x \le C_0^{u,z} + \widetilde{c}^H|\Omega| + \tfrac{1}{2}\int_\Omega \vartheta^0\,\mathrm{d}x - \tfrac{1}{2}\int_{\mathcal{Q}_{\widetilde{\tau}}}\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u}))\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_{\widetilde{\tau}}}\ell\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By using (3.11b), (3.14a), (3.15a), (3.15b) and Cauchy-Schwarz's and Korn's inequalities, we find

$$\begin{aligned}
&\tfrac{c^{\mathbb{E}}}{2}\|\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} + c^H\|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbb{L}}}{2}\int_0^{\widetilde{\tau}}\|\mathrm{e}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{c^{\mathbb{M}}}{2}\int_0^{\widetilde{\tau}}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\\
&+\tfrac{1}{2}\int_\Omega\vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x \le C_0^{u,z} + \widetilde{c}^H|\Omega| + \tfrac{1}{2}\int_\Omega\vartheta^0\,\mathrm{d}x + \tfrac{c^{\mathbb{L}}}{8}\int_0^{\widetilde{\tau}}\|\mathrm{e}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\\
&+\tfrac{c^{\mathbb{L}}C^{\mathsf{Korn}}}{8}\int_0^{\widetilde{\tau}}\|\dot{u}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{2}{c^{\mathbb{L}}C^{\mathsf{Korn}}}\int_0^{\widetilde{\tau}}\|\ell\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{9\alpha^2}{2c^{\mathbb{L}}}\int_0^{\widetilde{\tau}}\|\theta\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

We estimate the last term by using (3.18), we have

$$\int_0^{\widetilde{\tau}}\|\theta\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t = \int_{\mathcal{Q}_{\widetilde{\tau}}}|\zeta(\vartheta)|^2\,\mathrm{d}x\,\mathrm{d}t \le \big(\tfrac{\beta_1}{c^c}\big)^{\frac{2}{\beta}}\int_{\mathcal{Q}_{\widetilde{\tau}}}|\vartheta|^{\frac{2}{\beta}}\,\mathrm{d}x\,\mathrm{d}t,$$

for all $\beta\in[1,\beta_1]$. Since $\beta_1\ge 2$, we get

$$\begin{aligned}
&\tfrac{c^{\mathbb{E}}}{2}\|\mathrm{e}(u(\cdot,\widetilde{\tau}))-\mathbb{B}z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} + c^H\|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} + \tfrac{c^{\mathbb{L}}}{4}\int_0^{\widetilde{\tau}}\|\mathrm{e}(\dot{u})\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t + \tfrac{c^{\mathbb{M}}}{2}\int_0^{\widetilde{\tau}}\|\dot{z}\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t\\
&+\tfrac{1}{2}\int_\Omega\vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x \le C_0^{u,z}+\widetilde{c}^H|\Omega|+\tfrac{1}{2}\int_\Omega\vartheta^0\,\mathrm{d}x+\tfrac{2}{c^{\mathbb{L}}C^{\mathsf{Korn}}}\int_0^{\widetilde{\tau}}\|\ell\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t+\tfrac{9\alpha^2\beta_1}{2c^{\mathbb{L}}c^c}\int_0^{\widetilde{\tau}}\|\vartheta\|_{\mathrm{L}^1(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Since $\vartheta(x,t)\ge 0$ almost everywhere on $\mathcal{Q}_\tau$ and $\ell\in\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, we may conclude that (7.1) holds by using Grönwall's lemma.

Finally, we use the techniques introduced by Boccardo and Gallouët in [BoG89, BoG92] to obtain the last estimate for the enthalpy ϑ. More precisely, let us consider the function $h:\mathbb{R}^+\to[0,1]$ defined by

$$\forall s\ge 0:\ h(s)\overset{\mathsf{def}}{=}1-\tfrac{1}{(1+s)^\gamma}\quad \text{with } \gamma>0.$$

This function h belongs to $\mathrm{C}^1(\mathbb{R}^+;\mathbb{R})$ and is γ-Lipschitz continuous on $\mathbb{R}^+$. We define $\mathcal{H}$ as the unique primitive of h, which vanishes at $s=0$, i.e., we have

$$\forall s\in\mathbb{R}^+:\ \mathcal{H}(s)\overset{\mathsf{def}}{=}\int_0^s h(\sigma)\,\mathrm{d}\sigma.$$

We infer that $h(\vartheta)\in\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega))$ and thus may be used as a test-function in (3.7c). We obtain

$$\begin{aligned}
&\int_{\mathcal{Q}_{\widetilde{\tau}}}\dot{\vartheta}h(\vartheta)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{Q}_{\widetilde{\tau}}}\tfrac{\gamma}{(1+\vartheta)^{1+\gamma}}\widetilde{\kappa}^c\nabla\vartheta:\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{\mathcal{Q}_{\widetilde{\tau}}}(\mathbb{L}\mathrm{e}(\dot{u}):\mathrm{e}(\dot{u})+\mathbb{M}\dot{z}.\dot{z}+\Psi(\dot{z})+\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})))h(\vartheta)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Observing that $h(\vartheta) \in [0,1)$ almost everywhere in $\mathcal{Q}_{\widetilde{\tau}}$, the right hand side of this equation may be estimated by using the same tricks as in Theorem 6.2. Indeed, we have

$$
\begin{aligned}
&\Big|\int_{\mathcal{Q}_{\widetilde{\tau}}} (\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\mathbb{M}\dot{z}.\dot{z}{+}\Psi(\dot{z}){+}\alpha\theta\mathrm{tr}(\mathrm{e}(\dot{u})))h(\vartheta)\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\le \int_{\mathcal{Q}_{\widetilde{\tau}}} \big(\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\mathbb{M}\dot{z}.\dot{z}{+}\Psi(\dot{z}){+}3\alpha|\theta||\mathrm{e}(\dot{u})|\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{\mathcal{Q}_{\widetilde{\tau}}} \big(\tfrac{3}{2}\mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}){+}\mathbb{M}\dot{z}.\dot{z}{+}\Psi(\dot{z}){+}\tfrac{9\alpha^2}{2c^{\mathbb{L}}}|\theta|^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \tfrac{3}{2}C^{\mathbb{L}}\|\mathrm{e}(\dot{u})\|^2_{\mathrm{L}^2(0,\widetilde{\tau};\mathrm{L}^2(\Omega))} + \big(C^{\mathbb{M}}{+}\tfrac{C^\Psi}{2}\big)\|\dot{z}\|^2_{\mathrm{L}^2(0,\widetilde{\tau};\mathrm{L}^2(\Omega))} + \tfrac{C^\Psi}{2}T|\Omega| + \tfrac{9\alpha^2\beta_1}{2c^{\mathbb{L}}c^c}\widetilde{\tau}\|\vartheta\|_{\mathrm{L}^\infty(0,\widetilde{\tau};\mathrm{L}^1(\Omega))}.
\end{aligned}
$$

Using the previous estimate, we infer that, possibly modifying the generic constant $\widetilde{C}$, we have

$$
\int_\Omega \mathcal{H}(\vartheta(\cdot,\widetilde{\tau}))\,\mathrm{d}x + \gamma c^{\kappa^c}\int_{\mathcal{Q}_{\widetilde{\tau}}} \tfrac{|\nabla\vartheta|^2}{(1+\vartheta)^{1+\gamma}}\,\mathrm{d}x\,\mathrm{d}t \le \widetilde{C} + \int_\Omega \mathcal{H}(\vartheta^0)\,\mathrm{d}x.
$$

Let $1 \le r < 2$. We can estimate $\int_{\mathcal{Q}_{\widetilde{\tau}}}|\nabla\vartheta|^r\,\mathrm{d}x\,\mathrm{d}t$ by combining the previous inequality with Hölder's and Gagliardo-Nirenberg's inequalities. More precisely, we have

$$
\begin{aligned}
&\int_{\mathcal{Q}_{\widetilde{\tau}}} |\nabla\vartheta|^r\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{Q}_{\widetilde{\tau}}} \tfrac{|\nabla\vartheta|^r}{(1+\vartheta)^{\frac{(1+\gamma)r}{2}}}(1{+}\vartheta)^{\frac{(1+\gamma)r}{2}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \Big(\int_{\mathcal{Q}_{\widetilde{\tau}}} \tfrac{|\nabla\vartheta|^2}{(1+\vartheta)^{(1+\gamma)}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{r}{2}}\Big(\int_{\mathcal{Q}_{\widetilde{\tau}}}(1{+}\vartheta)^{\frac{(1+\gamma)r}{2-r}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2-r}{2}} \\
&\le \tfrac{1}{(\gamma c^{\kappa^c})^{\frac{r}{2}}}\Big(\widetilde{C}{+}\int_\Omega \mathcal{H}(\vartheta^0)\,\mathrm{d}x\Big)^{\frac{r}{2}}\Big(\int_0^{\widetilde{\tau}}\|1{+}\vartheta(t,\cdot)\|^{\frac{(1+\gamma)r}{2-r}}_{\mathrm{L}^{(1+\gamma)r/(2-r)}(\Omega)}\,\mathrm{d}t\Big)^{\frac{2-r}{2}},
\end{aligned}
\tag{7.6}
$$

and, for almost every $t \in [0,\tau]$, we have

$$
\begin{aligned}
\|1{+}\vartheta(\cdot,t)\|_{\mathrm{L}^{(1+\gamma)r/(2-r)}(\Omega)} &\le C_{\mathrm{GN}}\big(\|1{+}\vartheta(\cdot,t)\|_{\mathrm{L}^1(\Omega)}{+}\|\nabla\vartheta(\cdot,t)\|_{\mathrm{L}^r(\Omega)}\big)^\mu\|1{+}\vartheta(\cdot,t)\|^{1-\mu}_{\mathrm{L}^1(\Omega)} \\
&\le C_{\mathrm{GN}}(|\Omega| + \widetilde{C})^{1-\mu}\big(|\Omega|{+}\widetilde{C}{+}\|\nabla\vartheta(\cdot,t)\|_{\mathrm{L}^r(\Omega)}\big)^\mu,
\end{aligned}
\tag{7.7}
$$

with a constant $C_{\mathrm{GN}} > 0$ depending only on r, γ and μ and

$$
\tfrac{2-r}{(1+\gamma)r} \ge \mu\big(\tfrac{1}{r}{-}\tfrac{1}{3}\big) + 1 - \mu, \quad 0 < \mu \le 1.
$$

We choose $\mu = \frac{2-r}{1+\gamma}$. Using (7.1), we infer that there exists a constant $C_{r,\gamma} > 0$, depending only on r, γ and $\widetilde{C}$ such that

$$
\begin{aligned}
&\Big(\int_0^{\widetilde{\tau}}\|1{+}\vartheta(\cdot,t)\|^{\frac{(1+\gamma)r}{2-r}}_{\mathrm{L}^{(1+\gamma)r/(2-r)}(\Omega)}\,\mathrm{d}t\Big)^{\frac{2-r}{2}} \\
&\le \Big(\int_0^{\widetilde{\tau}}\big(C_{\mathrm{GN}}(|\Omega|{+}\widetilde{C})^{1-\mu}(|\Omega|{+}\widetilde{C}{+}\|\nabla\vartheta(\cdot,t)\|_{\mathrm{L}^r(\Omega)})^\mu\big)^{\frac{(1+\gamma)r}{2-r}}\,\mathrm{d}t\Big)^{\frac{2-r}{2}} \\
&\le C_{r,\gamma}\big(1{+}\|\nabla\vartheta\|^{r(1-\frac{r}{2})}_{\mathrm{L}^r(0,\widetilde{\tau};\mathrm{L}^r(\Omega))}\big).
\end{aligned}
\tag{7.8}
$$

We insert (7.8) into (7.6) and we obtain

$$
\frac{\|\nabla\vartheta\|^r_{\mathrm{L}^r(0,\widetilde{\tau};\mathrm{L}^r(\Omega))}}{1{+}\|\nabla\vartheta\|^{r(1-\frac{r}{2})}_{\mathrm{L}^r(0,\widetilde{\tau};\mathrm{L}^r(\Omega))}} \le \tfrac{C_{r,\gamma}}{(\gamma c^{\kappa^c})^{\frac{r}{2}}}\Big(\widetilde{C}{+}\int_\Omega \mathcal{H}(\vartheta^0)\,\mathrm{d}x\Big)^{\frac{r}{2}}.
$$

Since the mapping $s \mapsto \frac{s^{\frac{2}{2-r}}}{1+s}$ is increasing on $\mathbb{R}^+$ and tends to $+\infty$ as s tends to $+\infty$, we may conclude that $\|\nabla\vartheta\|^{r(1-\frac{r}{2})}_{\mathrm{L}^r(0,\widetilde{\tau};\mathrm{L}^r(\Omega))}$ is bounded by a constant depending only on $\widetilde{C}$ and on the choice of r and γ. The two conditions $\frac{2-r}{(1+\gamma)r} \geq \mu\big(\frac{1}{r}-\frac{1}{3}\big) + 1 - \mu$ with $\mu \overset{\text{def}}{=} \frac{2-r}{1+\gamma}$ and $\gamma > 0$ imply that $r \leq \frac{5}{4} - \frac{3}{4}\gamma < \frac{5}{4}$. Hence, we can choose any $r \in [1, \frac{5}{4})$, with a corresponding choice of γ as $\gamma = \frac{4}{3}\big(\frac{5}{4}-r\big) > 0$, and the conclusion follows. $\square$

Remark 7.2 *We can observe that, whenever $c^H > 0$, the symmetry and the coercivity properties of $\mathbb{E}$ imply that there exist two real numbers $C_1, C_2 > 0$, depending only on $\mathbb{B}_0$, B_1, c^H, $c^{\mathbb{E}}$ and $\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}$, such that for almost every $x \in \Omega$*

$$\forall \mathrm{e} \in \mathbb{R}^{3\times 3}_{\mathsf{sym}},\ \forall z \in \mathcal{Z}:\ \mathbb{E}(\mathrm{e}-\mathbb{B}z){:}(\mathrm{e}-\mathbb{B}z) + 2c^H|z|^2 \geq C_1(|\mathrm{e}|^2+|z|^2) - C_2.$$

In this case, possibly modifying $\widetilde{C}$, we will also get

$$\forall \widetilde{\tau} \in [0,\tau]:\ \|u(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)} \leq \widetilde{C}.$$

Once again using Gagliardo-Nirenberg's inequality, we get

$$\begin{aligned}\|\vartheta\|^{\widetilde{\alpha}}_{\mathrm{L}^{\widetilde{\alpha}}(0,\tau;\mathrm{L}^{\widetilde{\alpha}}(\Omega))} &= \int_0^\tau \|\vartheta(\cdot,t)\|^{\widetilde{\alpha}}_{\mathrm{L}^{\widetilde{\alpha}}(\Omega))}\,\mathrm{d}t \\ &\leq C_{\mathrm{GN}} \int_0^\tau \|\vartheta(\cdot,t)\|^{\widetilde{\alpha}(1-\widetilde{\mu})}_{\mathrm{L}^1(\Omega)} (\|\vartheta(\cdot,t)\|_{\mathrm{L}^1(\Omega)} + \|\nabla\vartheta(\cdot,t)\|_{\mathrm{L}^r(\Omega)})^{\widetilde{\alpha}\widetilde{\mu}}\,\mathrm{d}t,\end{aligned}$$

with

$$\tfrac{1}{\widetilde{\alpha}} \geq \widetilde{\mu}\big(\tfrac{1}{r}-\tfrac{1}{3}\big) + 1 - \widetilde{\mu}, \quad 0 < \widetilde{\mu} \leq 1, \quad \text{and} \quad \widetilde{\alpha}\widetilde{\mu} \leq r, \quad 1 \leq r < \tfrac{5}{4}.$$

Using the results of the previous proposition, we get a global a priori estimate for ϑ in $\mathrm{L}^{\widetilde{\alpha}}(0,\tau;\mathrm{L}^{\widetilde{\alpha}}(\Omega))$, independently of τ. A possible choice for $\widetilde{\alpha}$ is obtained when $\frac{1}{\widetilde{\alpha}} = \widetilde{\mu}\big(\frac{1}{r}-\frac{1}{3}\big) + 1 - \widetilde{\mu} = \frac{\widetilde{\mu}}{r}$, i.e., $\widetilde{\mu} = \frac{3}{4}$, which yields $\widetilde{\alpha} = \frac{4r}{3} \in \big[\frac{4}{3}, \frac{5}{3}\big)$. With (3.18) we infer that, for any $\beta_1 > \frac{12}{5}$, we can choose $r \in [1, \frac{5}{4})$ such that $\theta = \zeta(\vartheta) \in \mathrm{L}^{\widetilde{q}}(0,\tau;\mathrm{L}^{\widetilde{p}}(\Omega))$ with $\widetilde{q} = \widetilde{p} = \beta_1\widetilde{\alpha} = \beta_1\frac{4r}{3} > 4$. Moreover there exists a generic constant $C_{\vartheta,r} > 0$, depending only on the data and r such that

$$\|\zeta(\vartheta)\|_{\mathrm{L}^{\widetilde{q}}(0,\tau;\mathrm{L}^{\widetilde{p}}(\Omega))} = \|\theta\|_{\mathrm{L}^{\widetilde{q}}(0,\tau;\mathrm{L}^{\widetilde{p}}(\Omega))} \leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta_1}} \|\vartheta\|^{\frac{1}{\beta_1}}_{\mathrm{L}^{\widetilde{\alpha}}(0,\tau;\mathrm{L}^{\widetilde{\alpha}}(\Omega))} \leq C_{\vartheta,r},$$

for any solution $(u, z, \theta = \zeta(\vartheta))$ of problem (3.2)–(3.4) defined on $[0,\tau]$, $\tau \in (0,T]$.

Now let

$$\bar{R}^\theta = C_{\vartheta,r}|\Omega|^{\frac{\widetilde{p}-4}{4\widetilde{p}}}.$$

By using the notations of Section 6, let

$$\begin{aligned}&\bar{R}^f \overset{\text{def}}{=} R^f(R^0, \bar{R}^\theta, \|\ell\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}), \quad \bar{R}^\vartheta_\infty \overset{\text{def}}{=} C\big(T^{\frac{\widetilde{q}-4}{2\widetilde{q}}}\bar{R}^f + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big),\\ &\bar{R}^\vartheta \overset{\text{def}}{=} T^{\frac{2}{\widetilde{q}}}\bar{R}^\vartheta_\infty + 1.\end{aligned}$$

Then, we let $\bar{q} = \frac{\widetilde{q}}{2}$ and we may deduce from the results of Section 6 that there exists $\tau \in (0,T]$ such that $\phi^{\widetilde{\vartheta},\vartheta}_\tau$ admits a fixed point in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0,\bar{R}^\vartheta)$. Let us define

$$\bar{\tau} \overset{\text{def}}{=} \sup\big\{\tau \in (0,T] : \phi^{\widetilde{\vartheta},\vartheta}_\tau \text{ admits a fixed point in } \bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0,\bar{R}^\vartheta)\big\} \in (0,T].$$

It is clear that problem (3.7)–(3.9) admits a global solution if and only if $\bar{\tau} = T$. Let us prove this identity by a contradiction argument. First, we assume that $\bar{\tau} \in (0, T)$ and we choose $\epsilon \in (0, \bar{\tau})$. The definition of $\bar{\tau}$ implies that there exists $\tau \in (\bar{\tau}-\epsilon, \bar{\tau}]$ such that $\phi_\tau^{\widetilde{\vartheta},\vartheta}$ admits a fixed point $\vartheta = \phi_\tau^{\widetilde{\vartheta},\vartheta}(\vartheta)$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)$, i.e., the problem (3.7)–(3.9) possesses a solution (u, z, ϑ) defined on $[0, \tau]$. By combining the results of Proposition 7.1 together with the results of Section 6, it is easy to get $\|\theta\|_{\mathrm{L}^{\widetilde{q}}(0,\tau;\mathrm{L}^4(\Omega))} \leq \bar{R}^\theta$ with $\theta = \zeta(\vartheta)$ and $\vartheta \in \mathrm{L}^\infty(0, \tau; \mathrm{L}^2(\Omega))$ with $\|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \leq \bar{R}^\vartheta_\infty$.

Define $\widetilde{\tau} \in (0, T-\tau]$ and $\widetilde{R}^\vartheta \overset{\text{def}}{=} ((\bar{R}^\vartheta)^{\bar{q}} - T(\bar{R}^\vartheta_\infty)^{\bar{q}})^{\frac{1}{\bar{q}}} > 0$. For any $\widetilde{\vartheta} \in \bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$, we define

$$\widetilde{\vartheta}_{\text{ext}}(x,t) \overset{\text{def}}{=} \begin{cases} \vartheta(x,t) & \text{if } (x,t) \in \Omega \times [0,\tau], \\ \widetilde{\vartheta}(x,t) & \text{if } (x,t) \in \Omega \times (\tau, \tau+\widetilde{\tau}], \\ 0 & \text{if } (x,t) \in \Omega \times (\tau+\widetilde{\tau}, T]. \end{cases}$$

Clearly, we have

$$\begin{aligned} \|\widetilde{\vartheta}_{\text{ext}}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^2(\Omega))} &= \|\vartheta\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))} + \|\widetilde{\vartheta}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))} \\ &\leq \tau(\bar{R}^\vartheta_\infty)^{\bar{q}} + (\widetilde{R}^\vartheta)^{\bar{q}} \leq (\bar{R}^\vartheta)^{\bar{q}}, \end{aligned}$$

and the mapping $\widetilde{\vartheta} \mapsto \widetilde{\vartheta}_{\text{ext}}$ is a contraction on $\mathrm{L}^{\bar{q}}(\tau, \tau+\widetilde{\tau}; \mathrm{L}^2(\Omega))$. Furthermore the definition of $\phi^{\widetilde{\vartheta},\vartheta}$ gives that the restriction of $\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})$ on $[0, \tau]$ coincide with $\phi_\tau^{\widetilde{\vartheta},\vartheta}(\vartheta) = \vartheta$ and we define $\widetilde{\phi}_{\widetilde{\tau}}^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta})$ as the restriction of $\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})$ to $[\tau, \tau+\widetilde{\tau}]$. Note that the estimates obtained in Section 6 lead to $\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}}) \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega))$ and

$$\|\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \leq C\big(T^{\frac{\widetilde{q}-4}{2\widetilde{q}}} R^f(R^0, \widetilde{R}^\theta, \|\ell\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}) + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big),$$

with $\widetilde{R}^\theta \overset{\text{def}}{=} \big((\bar{R}^\theta)^{\widetilde{q}} + \big(\frac{\beta_1}{c^c}\big)^{\frac{\widetilde{q}}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1}\widetilde{q}} T^{\frac{\beta_1-2}{\beta_1}} (\widetilde{R}^\vartheta)^{\frac{\widetilde{q}}{\beta_1}}\big)^{\frac{1}{\widetilde{q}}}$. Indeed, let us denote $\widetilde{\theta} = \zeta(\widetilde{\vartheta}_{\text{ext}})$. We have

$$\widetilde{\theta}(x,t) \overset{\text{def}}{=} \begin{cases} \zeta(\vartheta(x,t)) = \theta(x,t) & \text{if } (x,t) \in \Omega \times [0,\tau], \\ \zeta(\widetilde{\vartheta}(x,t)) & \text{if } (x,t) \in \Omega \times (\tau, \tau+\widetilde{\tau}], \\ \zeta(0) = 0 & \text{if } (x,t) \in \Omega \times (\tau+\widetilde{\tau}, T]. \end{cases}$$

Since $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(\tau, \tau+\widetilde{\tau}; \mathrm{L}^2(\Omega))$ we have $\zeta(\widetilde{\vartheta}) \in \mathrm{L}^{\widetilde{q}}(\tau, \tau+\widetilde{\tau}; \mathrm{L}^4(\Omega))$ and with (3.18)

$$\begin{aligned} \|\zeta(\widetilde{\vartheta})\|_{\mathrm{L}^{\widetilde{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^4(\Omega))} &\leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1}} \widetilde{\tau}^{\frac{\beta_1-2}{\beta_1\widetilde{q}}} \|\widetilde{\vartheta}\|^{\frac{1}{\beta_1}}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))} \\ &\leq \big(\tfrac{\beta_1}{c^c}\big)^{\frac{1}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1}} T^{\frac{\beta_1-2}{\beta_1\widetilde{q}}} (\widetilde{R}^\vartheta)^{\frac{1}{\beta_1}}. \end{aligned}$$

Hence

$$\|\widetilde{\theta}\|_{\mathrm{L}^{\widetilde{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^4(\Omega))} = \Big(\int_0^\tau \|\theta\|^{\widetilde{q}}_{\mathrm{L}^4(\Omega)} \,\mathrm{d}t + \int_\tau^{\tau+\widetilde{\tau}} \|\zeta(\vartheta)\|^{\widetilde{q}}_{\mathrm{L}^4(\Omega)} \,\mathrm{d}t\Big)^{\frac{1}{\widetilde{q}}} \leq \widetilde{R}^\theta.$$

Now let $\widetilde{\tau}_0 > 0$ be such that

$$\widetilde{\tau}_0^{\frac{2}{\bar{q}}} C\big(T^{\frac{\widetilde{q}-4}{2\widetilde{q}}} R^f(R^0, \widetilde{R}^\theta, \|\ell\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))}) + \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}\big) \leq \widetilde{R}^\vartheta.$$

Note that the real number $\widetilde{\tau}_0$ does not depend on τ and/or $\bar{\tau}$ and for all $\widetilde{\tau} \in \big(0, \min(\widetilde{\tau}_0, T-\tau)\big]$, the mapping $\widetilde{\phi}_{\widetilde{\tau}}^{\widetilde{\vartheta},\vartheta}$ admits a fixed point $\widetilde{\vartheta}$ in the closed ball $\bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$. By construction of

$\widetilde{\phi}^{\widetilde{\vartheta},\vartheta}_{\widetilde{\tau}}$, the restriction of $\phi^{\widetilde{\vartheta},\vartheta}(\widetilde{\vartheta}_{\text{ext}})$ to $[0,\tau+\widetilde{\tau}]$ is also a fixed point of $\phi^{\widetilde{\vartheta},\vartheta}_{\tau+\widetilde{\tau}}$ in $\bar{B}_{\mathrm{L}^{\widetilde{q}}(0,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0,\bar{R}^{\vartheta})$. But we may choose $\epsilon \in \big(0,\min(\bar{\tau},\widetilde{\tau}_0,T-\bar{\tau})\big)$ such that $\tau+\widetilde{\tau} > \bar{\tau}-\epsilon+\widetilde{\tau} > \bar{\tau}$ for some choice of $\widetilde{\tau} \in \big(\epsilon,\min(\widetilde{\tau}_0,T-\bar{\tau})\big]$, which gives a contradiction with the definition of $\bar{\tau}$.

Hence we can conclude that $\bar{\tau}=T$ and consequently the global existence result for (3.2)–(3.4) follows.

Theorem 7.3 (Global existence result) *Assume that* (3.11)*,* (3.12)*,* (3.14)*,* (3.15)*,* (3.16) *and* (3.17) *hold and that $\beta_1 > \frac{12}{5}$. Then, for any $u^0 \in \mathrm{V}^4_0(\Omega;\mathbb{R}^3)$, $z^0 \in \mathrm{L}^4(\Omega;\mathcal{Z})$ and $\vartheta^0 \in \mathrm{L}^2(\Omega)$ such that* (6.2) *holds, the problem* (3.7)–(3.9) *admits a global solution (u,z,ϑ) such that $(u,z) \in \mathrm{W}^{1,\widetilde{q}}(0,T;\mathrm{V}^4_0(\Omega;\mathbb{R}^3) \times \mathrm{L}^4(\Omega;\mathcal{Z}))$ with $\widetilde{q} \in \big(4,\frac{5\beta_1}{3}\big)$, and $\vartheta \in \mathrm{W}^{\vartheta}$. Furthermore $\vartheta(t,x) > 0$ almost everywhere in $(0,T)\times\Omega$ and $(u,z,\theta=\zeta(\vartheta))$ is a solution of* (3.2)–(3.4) *on $[0,T]$.*

Remark 7.4 *If we consider a 2D setting, i.e., $\Omega \subset \mathbb{R}^2$, we can use the same arguments to prove a local existence result. Furthermore, global estimates can be obtained in the same way as in Proposition 7.1 with now $r \in [1,\frac{4}{3})$. Indeed, Gagliardo-Nirenberg's inequality in $\mathbb{R}^n$, $n=2$, leads to the condition*

$$\tfrac{2-r}{(1+\gamma)r} \geq \mu\left(\tfrac{1}{r}-\tfrac{1}{2}\right)+1-\mu, \quad 0<\mu\leq 1.$$

We can choose again $\mu = \frac{2-r}{1+\gamma}$ and we obtain $1 \leq r < \frac{4}{3}-\frac{2}{3}\gamma$. Reminding that $\gamma>0$, we may consider any value of $r \in [1,\frac{4}{3})$ with a corresponding $\gamma = \frac{3}{2}(\frac{4}{3}-r)$. Then we infer that, for any solution of the problem (3.7)–(3.9) *defined on $[0,\tau] \subset [0,T]$, the enthalpy ϑ is bounded in $\mathrm{L}^{\widetilde{\alpha}}(0,\tau,\mathrm{L}^{\widetilde{\alpha}}(\Omega))$ independently of τ, for any $\widetilde{\alpha}$ such that*

$$\tfrac{1}{\widetilde{\alpha}} \geq \widetilde{\mu}\left(\tfrac{1}{r}-\tfrac{1}{2}\right)+1-\widetilde{\mu}, \quad 0<\widetilde{\mu}\leq 1, \quad \textit{and} \quad \widetilde{\alpha}\widetilde{\mu}\leq r, \quad 1<r\leq\tfrac{4}{3}.$$

It follows that a possible choice for $\widetilde{\alpha}$ is given by

$$\tfrac{1}{\widetilde{\alpha}} = \widetilde{\mu}\left(\tfrac{1}{r}-\tfrac{1}{2}\right)+1-\widetilde{\mu} = \tfrac{\widetilde{\mu}}{r}$$

i.e. $\widetilde{\mu}=\frac{2}{3}$ and $\widetilde{\alpha}=\frac{3r}{2}\in[\frac{3}{2},2)$. Hence, for any $\beta_1>2$, there exists $r\in[1,\frac{4}{3})$ and $\widetilde{q}=\widetilde{p}=\beta_1\widetilde{\alpha}=\beta_1\frac{3r}{4}>4$ such that $\theta=\zeta(\vartheta)$ is bounded in $\mathrm{L}^{\widetilde{q}}(0,\tau,\mathrm{L}^{\widetilde{p}}(\Omega))$ independently of τ. Finally, by a contradiction argument, we may conclude with a global existence result under the weaker condition $\beta_1>2$.

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